A minimum Sobolev norm numerical technique for PDEs

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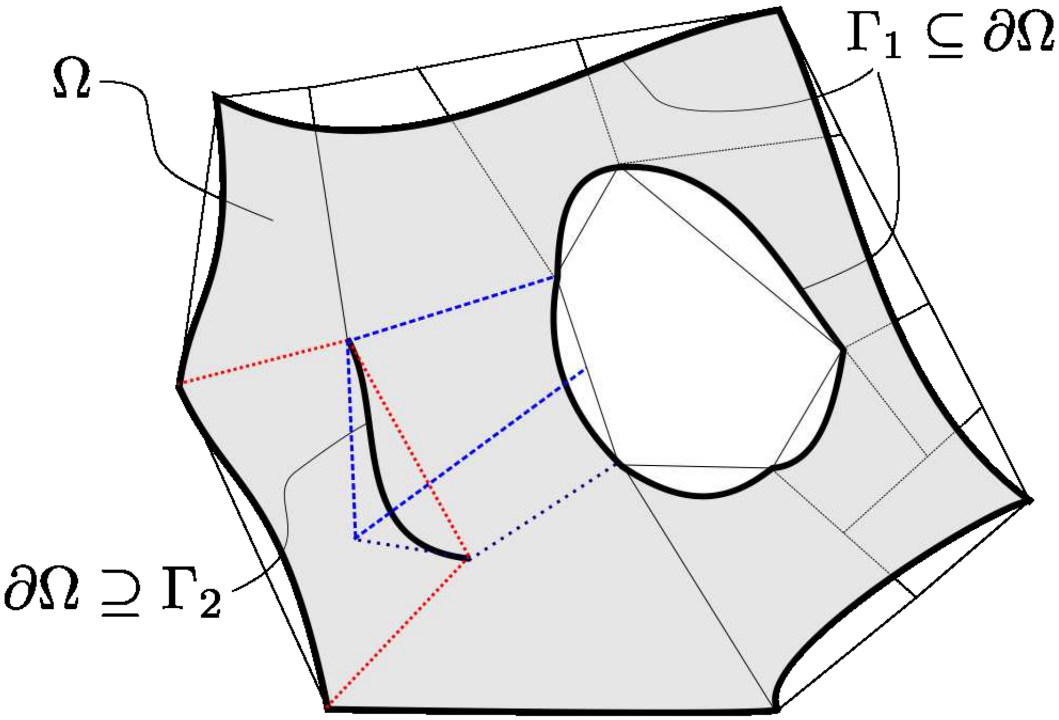
The Problem



- A vector-valued function u satisfies:

$$\begin{aligned} \mathcal{D}_1(\boldsymbol{x}, u) &= f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega \\ \mathcal{D}_2(\boldsymbol{x}, u) &= g(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega \end{aligned}$$

- $\mathcal{D}_i$ is a local linear differential operator with variable number of "rows"
- Find u numerically

	Problem	Code

Current `Octave` code assumes following form of PDE

$$\begin{aligned} \boldsymbol{A}_1\,\partial_1 u + \boldsymbol{A}_2\,\partial_2 u + \boldsymbol{B}u &= f, && \text{on } \Omega \\ \boldsymbol{C}u &= g, && \text{on } \partial\Omega \end{aligned}$$

where

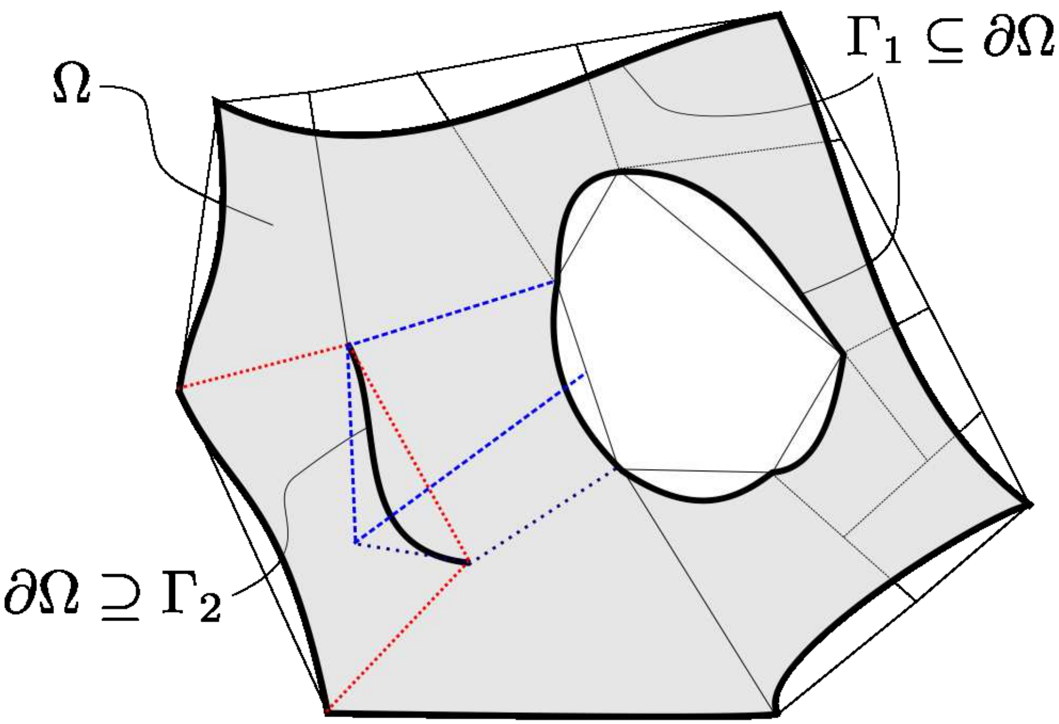
$$\begin{aligned} u &: \mathbb{R}^2 \to \mathbb{R}^q \\ f &: \mathbb{R}^2 \to \mathbb{R}^p \\ \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{B} &: \mathbb{R}^2 \to \mathbb{R}^{p \times q} \\ r &: \mathbb{R}^2 \to \mathbb{N} \\ \boldsymbol{C}(\boldsymbol{x}) &: \mathbb{R}^q \to \mathbb{R}^{r(\boldsymbol{x})} \end{aligned}$$

- u has q components
- f has p; not necessarily square
- Number of boundary conditions, $r(\boldsymbol{x})$, is allowed to vary
- No assumptions of homogeneity
- First-order form

First-order form

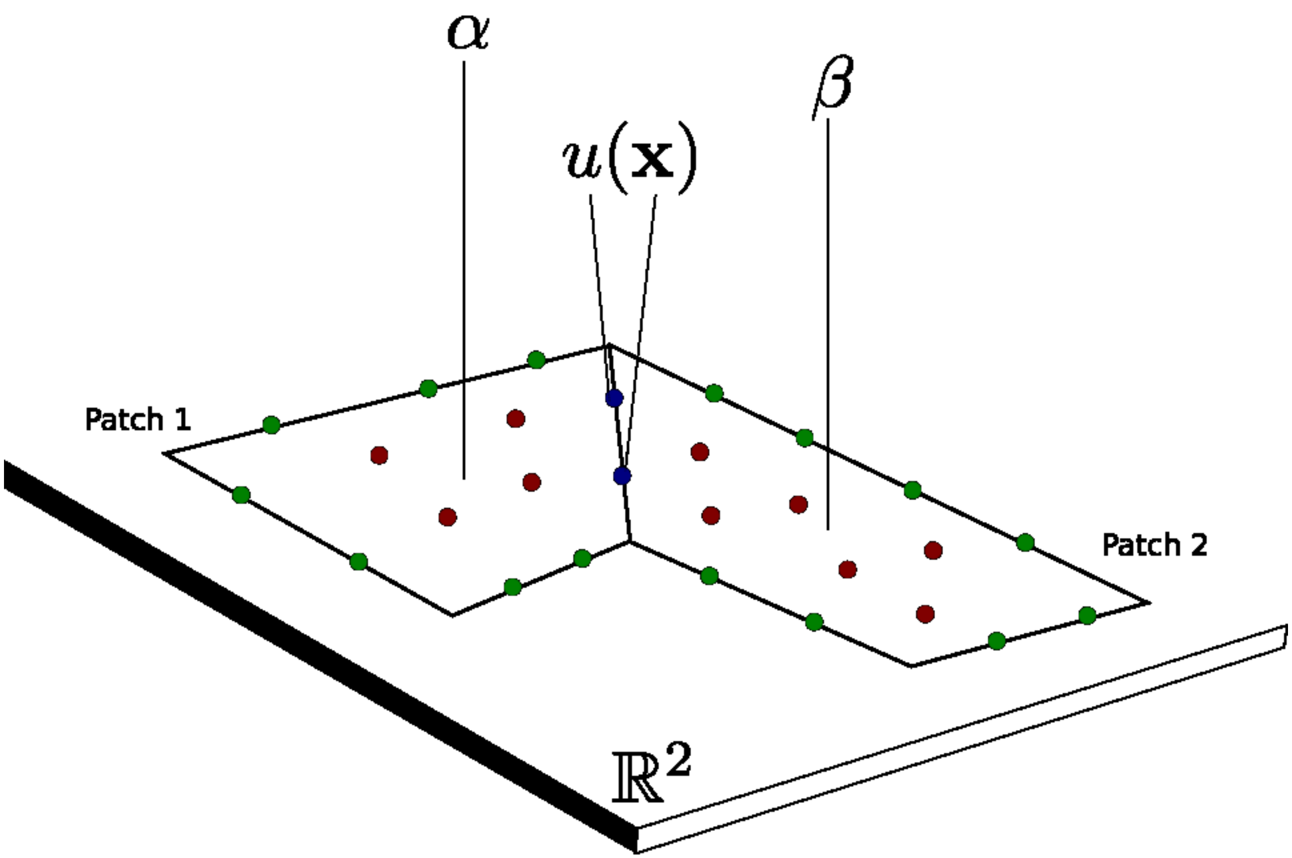
- Our method also works with higher-order derivatives
- FUD from previous attempts to use first-order form:
  - Missing boundary conditions for extra variables in first-order form
  - Mistaken assumption that discretized linear system must be square or skinny
  - Large memory foot-print problem for first-order form
  - Higher-order derivatives require more bits
  - No known numerical work on variable coefficient fourth-order PDEs
  - Seems to be missing from FEM, FD literature
- Fat is a great alternative

- $\overline{\Omega}$ is covered by strictly convex quadrilaterals called patches
- Patches can overlap
- Curved boundaries don't have to be approximated



On each patch we use modified 2D Chebyshev as basis

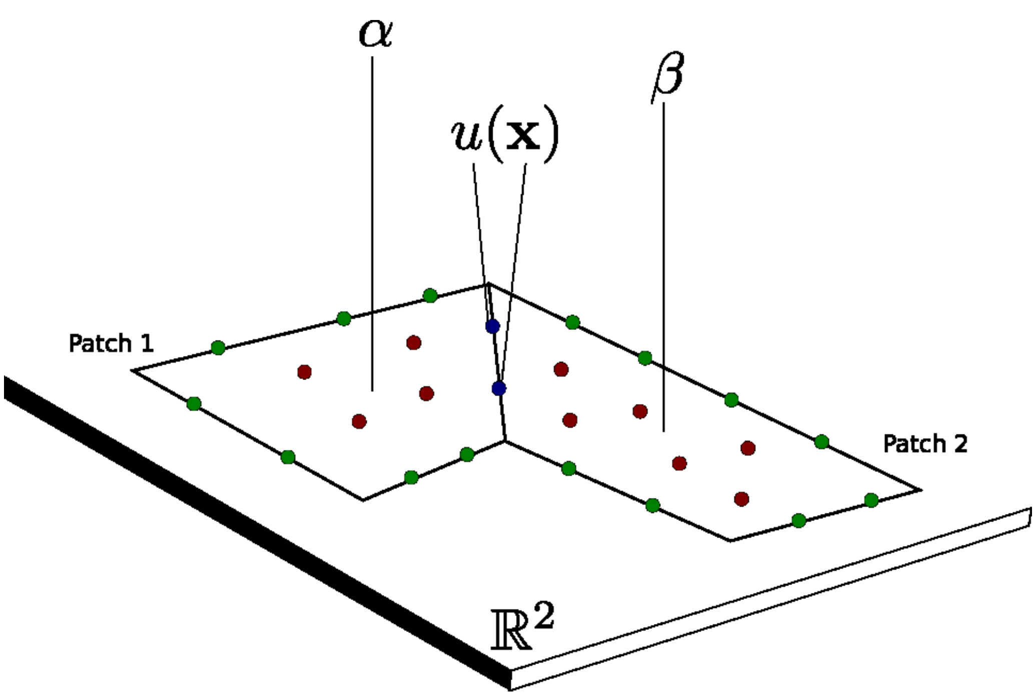
- $T_m(x) = \cos(m \cos^{-1} x)$ for $x \in [-1, 1]$
- $T_{\boldsymbol{m}}(\boldsymbol{x}) = T_{\boldsymbol{m}_1}(\boldsymbol{x}_1)\, T_{\boldsymbol{m}_2}(\boldsymbol{x}_2)$ for $\mathbb{N}^2$
- φ_P be the homography from patch P to $[-1, 1]^2$
- Bases on patch P: $T_{\boldsymbol{m}} \circ \varphi_P$ for $\boldsymbol{m} \in \mathbb{N}^2$
- Note that φ_P is from a strictly convex quadrilateral to the cube even if the patch overlaps a curved boundary
- No mapping problem like that for curved finite elements



$$u|_{\mathrm{Patch}_1} = \sum_{\boldsymbol{m}\in\mathbb{N}^2} \alpha_{\boldsymbol{m}} T_{\boldsymbol{m}} \circ \varphi_{\mathrm{Patch}_1}$$
$$u|_{\mathrm{Patch}_2} = \sum_{\boldsymbol{m}\in\mathbb{N}^2} \beta_{\boldsymbol{m}} T_{\boldsymbol{m}} \circ \varphi_{\mathrm{Patch}_2}$$

Disretization | Grid points

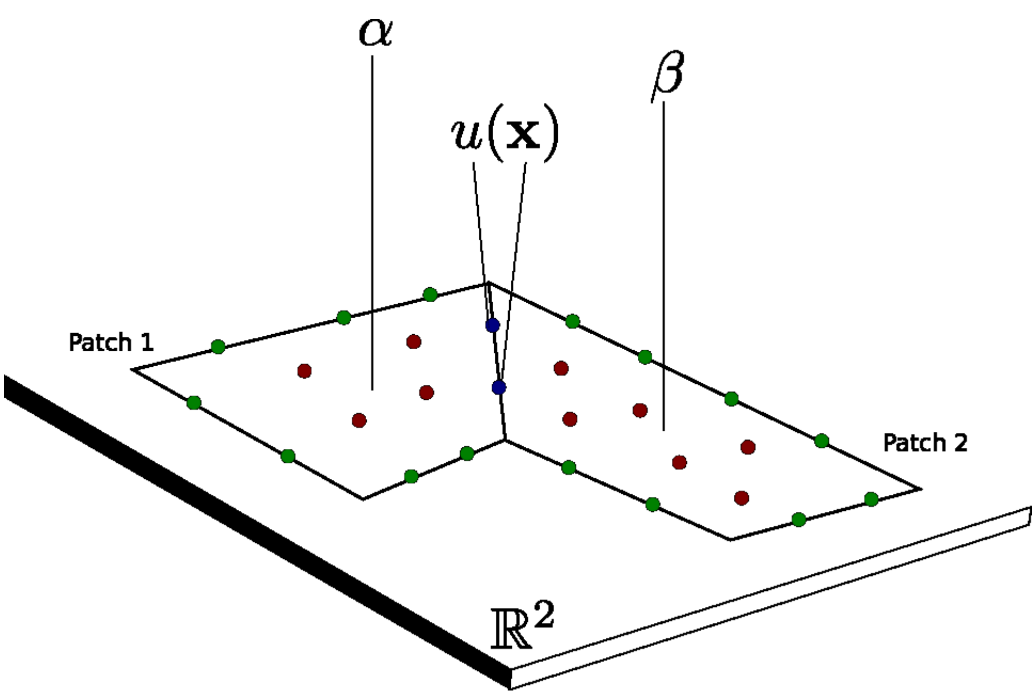
- We pick collocation as the discretization scheme
- Three types of grid points
  - Red points $\boldsymbol{x}_i$ interior to each patch and open set Ω
  - Green points $\boldsymbol{x}_i$ on boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$
  - Blue points $\boldsymbol{x}_i$ inside open set Ω and on interface edges shared between two patches



Discretization | Unknowns

- On each patch coefficients of Chebyshev expansions (α and β) are unknowns

- On blue interface points on each edge common to two patches u is an unknown



- For each patch collocate PDE at red interior points

$$\sum_{\boldsymbol{m}\in\mathbb{N}^2} (\boldsymbol{A}_1\,\partial_1 + \boldsymbol{A}_2\partial_2 + \boldsymbol{B})(T_{\boldsymbol{m}}\circ\varphi)(\boldsymbol{x}_i)\,\alpha_{\boldsymbol{m}} = f(\boldsymbol{x}_i)$$

- For each patch collocate boundary condition at green boundary points

$$\sum_{\boldsymbol{m}\in\mathbb{N}^2} \boldsymbol{C}(\boldsymbol{x}_i)(T_{\boldsymbol{m}}\circ\varphi)(\boldsymbol{x}_i)\,\alpha_{\boldsymbol{m}} = g(\boldsymbol{x}_i)$$

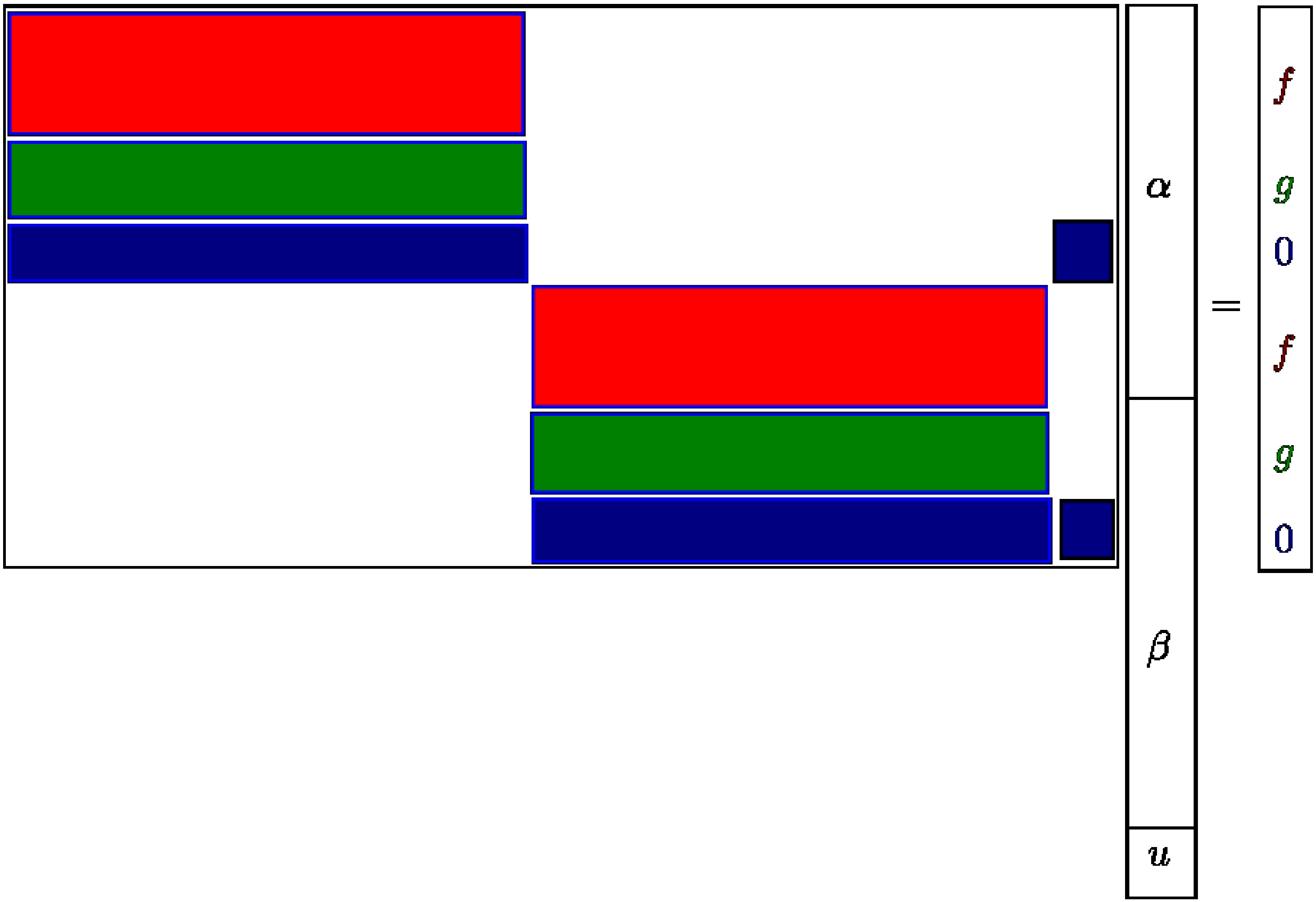
- For each patch collocate continuity conditions at blue interface points

$$\sum_{\boldsymbol{m}\in\mathbb{N}^2} (T_{\boldsymbol{m}}\circ\varphi)(\boldsymbol{x}_i)\,\alpha_{\boldsymbol{m}} = u(\boldsymbol{x}_i)$$

- Note that $u(\boldsymbol{x}_i)$ are the only unknowns connecting equations across patches

Assembled equations

The equations for the example problem:



Minimum Sobolev norm solution

- System is fat. Choose minimum norm solution. Which norm?
- Local s-Sobolev 2-norm on each patch

$$\|u|_{\mathrm{Patch}_1}\|_s^2 \equiv \sum_{\boldsymbol{m}\in\mathbb{N}^2} \|\alpha_{\boldsymbol{m}}\|^2 \cdot (1+\|\boldsymbol{m}\|^2)^s = \|D_s\alpha\|_2^2$$

where the standard Euclidean 2-norm uses

$$D_s = \mathrm{diag}\big((1+\|\boldsymbol{m}\|^2)^{s/2}\big)$$

- Global s-Sobolev norm

$$\|u\|_s^2 = \sum_{\mathrm{Patch}} \|u|_{\mathrm{Patch}_1}\|_s^2$$

- Large s leads to higher-order convergence. We use $s = 10$.
- Large s leads to severely ill-conditioned systems. We use special solvers.

Standard solver

- Write the equation as

$$\begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & 0 & A_{23} \\ 0 & A_{32} & 0 \\ 0 & A_{42} & A_{43} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ u_I \end{pmatrix} = \begin{pmatrix} fg \\ 0 \\ fg \\ 0 \end{pmatrix}$$

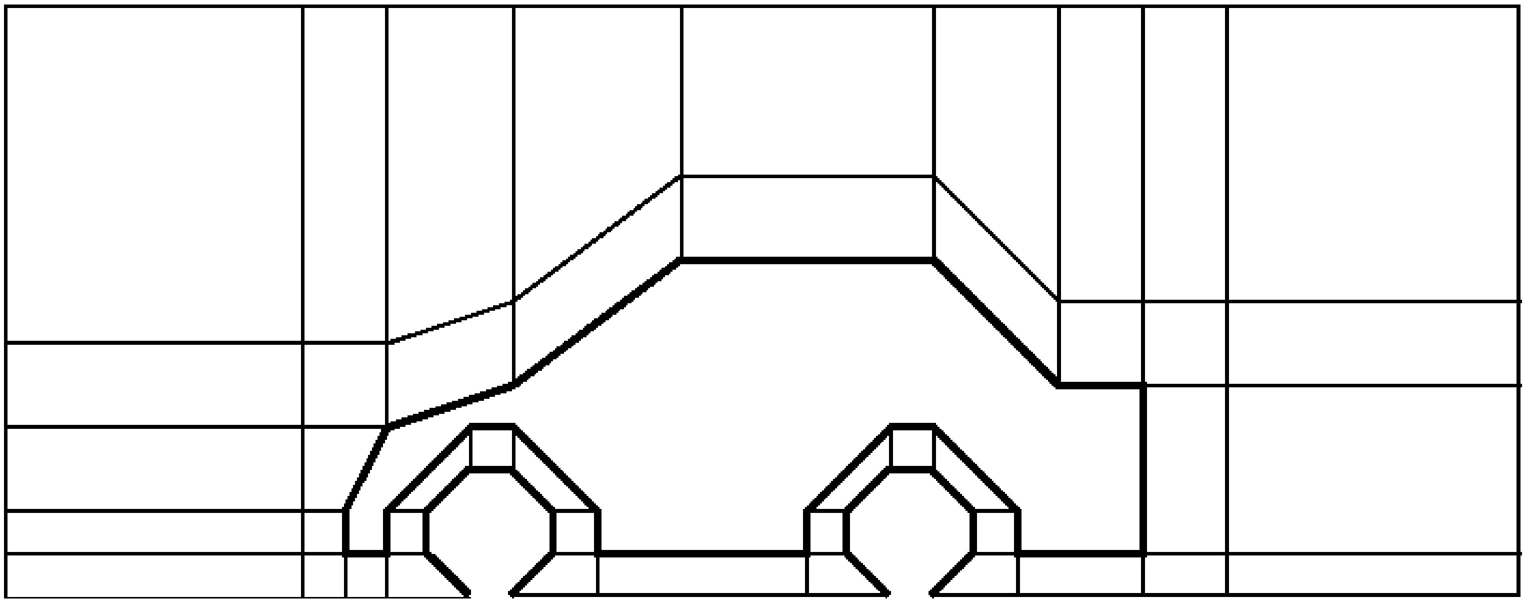
- For minimum s-Sobolev 2-norm solution insert D_s

$$\begin{pmatrix} A_{11} D_s^{-1} & 0 & 0 \\ A_{21} D_s^{-1} & 0 & A_{23} \\ 0 & A_{32}\, D_s^{-1} & 0 \\ 0 & A_{42}\, D_s^{-1} & A_{43} \end{pmatrix} \begin{pmatrix} D_s\, \alpha \\ D_s\, \beta \\ u_I \end{pmatrix} = \begin{pmatrix} fg \\ 0 \\ fg \\ 0 \end{pmatrix}$$

- Compute ordinary minimum 2-norm solution using standard sparse $L\,Q$ factorization.
- Convergence of solution (Golomb-Weinberger) can be established by standard compactness arguments using a variant of the Ascoli-Arzelia theorem with interpolation conditions.
- Assumptions include: existence & uniqueness of solution in appropriate Sobolev space, and linear independence of collocated equations.

Special solver

- For large s values standard solver fails numerically
- Similar problem for classical high-order methods
- Our problem has the form well-conditioned fat matrix times highly ill-conditioned diagonal matrix
- Matrix was made fat to make it well-conditioned (similar to compressive sensing)
- For such under-determined problems special work by [Stewart], [Hough & Vavasis], [Gu], [Castro-Gonzalez, Ceballos, Dopico & Molera], [Higham], etc.
- Special two-sided orthogonal decomposition with complete pivoting
- Extension by us to sparse case; also greatly reduces memory consumption
- Used in all numerical experiments
- Truncation of expansion requires sophisticated analysis [Chandrasekaran & Mhaskar, JCP, 2013]



- Large domain $\subseteq [0, 36] \times [0, 14]$
- Outer boundary is not rectangle; includes wheels
- Covered by 45 patches
- p-convergence; so no refinement of mesh in these experiments

$$\begin{aligned}
\theta(\boldsymbol{x}) &= \frac{\boldsymbol{x}_1}{1+\boldsymbol{x}_2} \\
\lambda(\boldsymbol{x}) &= \frac{1+\boldsymbol{x}_2}{1+\boldsymbol{x}_1} \\
\mu(\boldsymbol{x}) &= \frac{1+\boldsymbol{x}_1}{1+\boldsymbol{x}_2} \\
\mathcal{A}(\cdot) &= \begin{pmatrix} \lambda\cos^2\theta + \mu\sin^2\theta & \frac{1}{2}(\mu-\lambda)\sin 2\theta \\ \frac{1}{2}(\mu-\lambda)\sin 2\theta & \mu\cos^2\theta + \lambda\sin^2\theta \end{pmatrix} \\
\omega_a(\boldsymbol{x}) &= \frac{1}{1+a\,(\boldsymbol{x}_1-\boldsymbol{x}_2^2)^2} \\
\rho_b(\boldsymbol{x}) &= (1+\|\boldsymbol{x}\|_2^2)^b
\end{aligned}$$

- $\mathcal{A}(\boldsymbol{x}) > 0$ whenever $\boldsymbol{x} > 0$
- $\mathcal{A}$ has variable eigenvalues and variable eigenvectors
- ω_a has singularities on a parabola in $\mathbb{C}^2$ whose distance to the real plane $\mathbb{R}^2$ is controlled by a
- ρ_b is not a polynomial or a rational for $b \notin \mathbb{Z}$

Coefficients of PDE in first-order form

$$
\begin{aligned}
\boldsymbol{A}_1 &= \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & 1 \end{pmatrix} \\
\boldsymbol{A}_2 &= \begin{pmatrix} \mathcal{A}_{21} & \mathcal{A}_{22} \\ -1 & 0 \end{pmatrix} \\
\boldsymbol{B} &= \begin{pmatrix} \mu\cos\theta - \lambda\sin\theta & \lambda\cos\theta + \mu\sin\theta \\ 0 & 0 \end{pmatrix} \\
u &= \begin{pmatrix} \omega_1 \\ \rho_{1/4} \end{pmatrix} && \text{known solution} \\
\boldsymbol{C} &= \tau^T && \text{tangential boundary conditions on outer rectangle} \\
\boldsymbol{C} &= \nu^T \mathcal{A} && \text{normal boundary conditions on car body}
\end{aligned}
$$

Grid size	Max rel. error	Compr. time (secs./patch)	Sparse solve time
0.62	1E-2	5	0.5
0.35	1E-3	78	2.3
0.24	3E-4	599	6.3

	Exterior of car	Variable coefficient scalar elliptic PDE

$$\nabla^T \mathcal{A} \nabla v + b^T \mathcal{A} \nabla v + cv = f_1$$

Coefficients in 3×3 first-order form:

$$\boldsymbol{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ \mathcal{A}_{11} & 0 & 0 \\ \mathcal{A}_{21} & 0 & 0 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ \mathcal{A}_{12} & 0 & 0 \\ \mathcal{A}_{22} & 0 & 0 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} c & b_1 & b_2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$b = \begin{pmatrix} \mu \cos\theta - \lambda \sin\theta \\ \lambda \cos\theta + \mu \sin\theta \end{pmatrix} \qquad c = -\sqrt{\lambda^2 + \mu^2}$$

$$f = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \qquad u = \begin{pmatrix} v \\ \mathcal{A}\nabla v \end{pmatrix} \qquad v = \omega_{1/10}$$

$$\boldsymbol{C} = \underset{\text{Dirichlet}}{(\,1 \;\; 0 \;\; 0\,)} \quad \text{or} \quad \underset{\text{Neumann}}{(\,0 \;\; \nu_1 \;\; \nu_2\,)} \quad \text{or} \quad \underset{\text{Tangential}}{(\,0 \;\; \tau_1 \;\; \tau_2\,)} \quad \text{or} \quad \underset{\text{Mixed}}{(\,* \;\; * \;\; *\,)}$$

Experimental results:

Grid size	Max rel. error	Compr. time (secs./patch)	Sparse solve time
0.62	4E-2	23	2.4
0.35	1E-3	413	13.4
0.24	8E-5	3192	39.5

- This includes error in (some linear combination of) derivatives of the solution

$E > 0$ is Young's modulus, $-1 < v < \frac{1}{2}$ is Poisson's ratio,

$$\mathcal{D} = \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1}{2}(1-2v) \end{pmatrix}, \qquad E = \lambda, \qquad v = \frac{\mu - 2\lambda}{2(\mu+\lambda)}$$

w is displacement, σ is elastic stress tensor, u is unknown,

$$\sigma = \mathcal{D} \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_2 \\ \partial_2 & \partial_1 \end{pmatrix} w, \qquad w = \begin{pmatrix} \omega_{1/10} \\ \rho_{3/4} \end{pmatrix}, \qquad u = \begin{pmatrix} w \\ \sigma \end{pmatrix} \in \mathbb{R}^5$$

First-order 5×5 form coefficients:

$$\boldsymbol{A}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \mathcal{D}_{11} & 0 & 0 & 0 & 0 \\ \mathcal{D}_{21} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{D}_{33} & 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \mathcal{D}_{12} & 0 & 0 & 0 \\ 0 & \mathcal{D}_{22} & 0 & 0 & 0 \\ \mathcal{D}_{33} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad f = \begin{pmatrix} -F_1 \\ -F_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

F is body force

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$ Displacement boundary condition

$$\boldsymbol{C} = \begin{pmatrix} 0 & 0 & \nu_1 & 0 & \nu_2 \\ 0 & 0 & 0 & \nu_2 & \nu_1 \end{pmatrix}$$ Traction boundary condition

We chose displacement boundary conditions everywhere.

Experimental results:

Grid size	Max rel. error	Compr. time (secs./patch)	Sparse solve time
0.73	9E-4	16	3.4
0.62	3E-4	39	5.8
0.53	1E-4	84	8.6
0.47	6E-5	169	12.8
0.42	3E-5	319	17.4
0.38	1E-5	551	23.0
0.35	5E-6	960	30.1

- This includes error in (some linear combination of) derivatives of the displacement (the elastic stress tensor)

b is base flow, w is deviation from base flow, p is pressure, v is viscosity coeff.

$$-\nabla p + v\nabla^T\nabla w + (b^T\nabla)w + (w^T\nabla)b = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad\qquad \nabla^T w = f_3$$

We chose

$$b = \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \qquad v = \frac{1}{10} \qquad w = \begin{pmatrix} \omega_{1/10} \\ \rho_{3/4} \end{pmatrix} \qquad p(\boldsymbol{x}) = \sin(\boldsymbol{x}_1 - \boldsymbol{x}_2)$$

Unknowns for 7×7 first-order form:

$$u = \begin{pmatrix} w \\ p \\ \nabla w_1 \\ \nabla w_2 \end{pmatrix}$$

Coefficients of 7×7 first-order form:

$$\boldsymbol{A}_1 = \begin{pmatrix} b_1 & 0 & -1 & v & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & 0 & v & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} b_2 & 0 & 0 & 0 & v & 0 & 0 \\ 0 & b_2 & -1 & 0 & 0 & 0 & v \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} \partial_1 b_1 & \partial_2 b_1 & 0 & 0 & 0 & 0 & 0 \\ \partial_1 b_2 & \partial_2 b_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Flow boundary conditions}$$

$$\boldsymbol{C} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Pressure boundary condition}$$

- Specified pressure on left and right outer vertical edges
- Specified flow everywhere else on boundary
- Note different number of boundary conditions on different parts of boundary

Experimental results:

Grid size	Max rel. error	Compr. time (secs./patch)	Sparse solve time
1.14	1E-3	3	2.4
0.89	4E-4	11	5.0
0.73	1E-4	35	9.1
0.62	7E-5	91	14.9
0.53	3E-5	205	23.0
0.47	7E-6	416	33.3

	Exterior of car	Variable coefficient scalar fourth-order elliptic PDE

$$\square = \begin{pmatrix} \partial_1^2 \\ \partial_1\partial_2 \\ \partial_2^2 \end{pmatrix} \qquad \square^T = \begin{pmatrix} \partial_1^2 & \partial_1\partial_2 & \partial_2^2 \end{pmatrix}$$

- $\mathcal{B}\colon \mathbb{R}^2 \to \mathbb{R}^{3\times 3}$ take values that are symmetric positive-definite matrices
- $\mathcal{C}\colon \mathbb{R}^2 \to \mathbb{R}^{3\times 2}$ and $\mathcal{C}\circ\nabla = \begin{pmatrix} \Sigma_j \mathcal{C}_{1j}\partial_j & \Sigma_j \mathcal{C}_{2j}\partial_j & \Sigma_j \mathcal{C}_{3j}\partial_j \end{pmatrix}$

PDE:

$$\square^T \mathcal{B}\square w + (\mathcal{C}\circ\nabla)\mathcal{B}\square w + d^T\mathcal{B}\square w + e^T\nabla w + cw = f_1$$

Bi-harmonic equation is a special case.
We chose

$$\mathcal{B} = \begin{pmatrix} 1 & \mu & 0 \\ \mu & 1+\mu^2 & \lambda \\ 0 & \lambda & 1+\lambda^2 \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \\ 1 & 1 \end{pmatrix} \quad d = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\mu \quad e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\lambda$$

$$c = \rho_{3/4} \qquad w = \omega_{1/100}$$

Unknowns for 9×9 first-order form:

$$u = \begin{pmatrix} w \\ \nabla w \\ \mathcal{B}\Box w \\ \begin{pmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_1 & 0 \\ 0 & 0 & \partial_2 \end{pmatrix} \mathcal{B}\Box w \end{pmatrix} \in \mathbb{R}^9$$

Dirichlet and Neumann boundary conditions everywhere

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu_1 & \nu_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Coefficients of 9×9 first-order form:

$$\boldsymbol{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \mathcal{C}_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}_{11} & \mathcal{B}_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}_{23} & \mathcal{B}_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}_{31} & \mathcal{B}_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{A}_2 = \begin{pmatrix} 0 & 0 & 0 & \mathcal{C}_{12} & \mathcal{C}_{22} & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{B}_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{B}_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{B}_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$
\boldsymbol{B} = \begin{pmatrix} c & e_1 & e_2 & d_1 & d_2 & d_3 & \mathcal{C}_{11} & \mathcal{C}_{21} & \mathcal{C}_{32} \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad f = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Experimental results:

Grid size	Max rel. error	Compr. time (secs./patch)	Sparse solve time
1.14	5E-3	6	5.0
0.89	2E-3	24	10.6
0.73	9E-4	73	19.1
0.62	4E-4	185	32.3
0.53	1E-4	422	48.4
0.47	8E-5	850	70.0

- This includes error in (some linear combination of) third derivatives

	Exterior of car	Poisson's equation in polar coordinates

$$\boldsymbol{x}_1^2 \partial_1^2 w + \boldsymbol{x}_1 \partial_1 w + \partial_2 w = f_1$$

Coefficients of 3×3 first-order form:

$$\boldsymbol{A}_1 = \begin{pmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_1^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad f = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix}$$

with solution

$$u = \begin{pmatrix} w \\ \nabla w \end{pmatrix} \qquad w = \boldsymbol{x}_1^{5/2} \omega_1(\boldsymbol{x})$$

Dirichlet boundary conditions everywhere $\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$

Experimental results:

Grid size	Max rel. error	Compr. time (secs./patch)	Sparse solve time
0.62	9E-2	10	1.5
0.53	4E-2	20	2.1
0.35	6E-3	212	7.2
0.32	3E-3	351	10.3
0.30	2E-3	577	13.4

	Exterior of car	Variable coefficient telegrapher's equation with 2-pt BC

- Vertical axis is cable
- Horizontal axis is time
- Along cable
  - V is voltage (unknown)
  - I is current (unknown)
  - C is capacitance
  - L is inductance
  - R is resistance
  - G is conductance
- Telegraphers equation in 2×2 first-order form is hyperbolic

$$\boldsymbol{A}_1 = \begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 0 & R \\ G & 0 \end{pmatrix} \qquad u = \begin{pmatrix} V \\ I \end{pmatrix}$$

- Rather than $V(0,\ \boldsymbol{x}_2)$ and $I(0,\ \boldsymbol{x}_2)$ as initial conditions we provide $V(0, \boldsymbol{x}_2)$ and $I(0, 36)$ as 2-point boundary conditions. Also V is provided at cable ends.
- Cable geometry and topology changes with time (ill-posed?)

We chose space and time-varying cable parameters

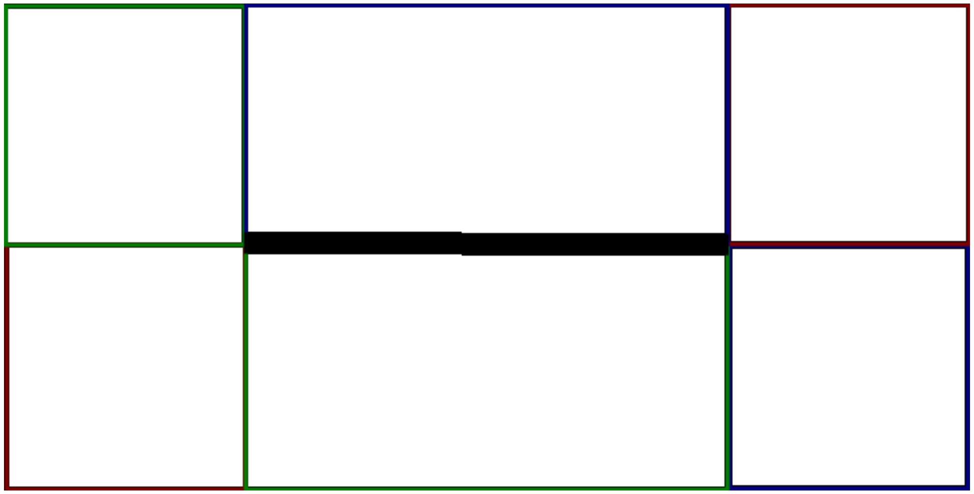
$$C = \lambda \qquad L = \mu \qquad R = \frac{\lambda}{2} + \mu \qquad G = \lambda + \frac{\mu}{2} \qquad V = \omega_{1/10} \qquad I = \rho_{3/4}$$

Experimental results:

Grid size	Max rel. error	Compr. time (secs./patch)	Sparse solve time
1.14	9E-5	4	0.1
0.89	7E-5	13	0.1
0.73	3E-5	36	0.2
0.62	2E-5	92	0.3
0.53	5E-5	201	0.5

- Last row shows a stall
- We used much longer Chebyshev expansions in this test than the other ones
- We conjecture that an even longer expansion will get out of the stall, or, the problem is ill-posed

Rectangle with slit



- 6 patches
- Thick line in the middle is a slit at $[-1, 1]$
- Outer rectangle is $[-2, 2] \times [-1, 1]$

Standard constant coefficient div-curl:

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

- $\iota^2 = -1,\ z = \boldsymbol{x}_1 + \iota \boldsymbol{x}_2$
- $(z^2 - 1)^{5/2} = u_R(\boldsymbol{x}) + \iota\, u_I(\boldsymbol{x})$ with branch cut on $[-1, 1]$ which is also the slit in the rectangle
- u_R is continuous across slit
- u_I is dis-continuous across slit
- We choose solution as

$$u = \begin{pmatrix} u_I \\ u_R \end{pmatrix}$$

. | | Rectangle with slit | Div-Curl | Single normal BC |
|---|---|---|---|

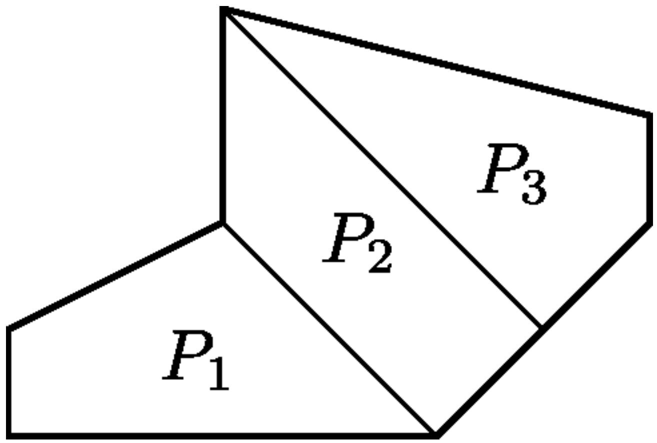
- Tangential boundary condition on outer boundary
- Single normal boundary condition on slit

Experimental results:

Grid size	Max rel. error	Compr. time (secs./patch)	Sparse solve time
0.29	6E-4	0.5	0.001
0.22	3E-4	1	0.001
0.18	2E-4	2	0.002
0.15	9E-5	4	0.002
0.09	1E-5	70	0.012
0.06	3E-6	533	0.035

- Normal boundary condition on outer boundary
- Double tangential boundary condition on slit; one as we approach slit from top, and one as we approach from bottom

Grid size	Max rel. error	Compr. time (secs./patch)	Sparse solve time
0.29	1E-3	0.5	0.001
0.22	5E-4	1	0.001
0.18	2E-4	2	0.002
0.15	1E-4	4	0.003
0.09	1E-5	67	0.012
0.06	4E-6	523	0.037



- Contained in $[0,3] \times [0,2]$
- Covered by three patches P_1, P_2, P_3
- P_2 is a trapezoid; this is exploited in constructing solution
- Coefficient will be dis-continuous across edges of P_2
- Solution will satisfy a jump condition on those edges

- Coefficients of PDE in first-order form

$$\boldsymbol{A}_1 = \begin{pmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} \\ 0 & 1 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} \mathcal{F}_{21} & \mathcal{F}_{22} \\ -1 & 0 \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} \mu\cos\theta - \lambda\sin\theta & \lambda\cos\theta + \mu\sin\theta \\ 0 & 0 \end{pmatrix}$$

- Jump condition at dis-continuity for this PDE

$$\begin{pmatrix} \nu^T \mathcal{F}_+ \\ \tau^T \end{pmatrix} u_+ = \begin{pmatrix} \nu^T \mathcal{F}_- \\ \tau^T \end{pmatrix} u_-$$

- $\mathcal{F}$ makes a complicated jump across edges of P_2

$$\mathcal{F}|_{P_1 \cup P_3} = \mathcal{A} \qquad \mathcal{F}\Big|_{P_2} = \begin{pmatrix} \mu\cos^2\theta + \lambda\sin^2\theta & \frac{1}{2}(\lambda - \mu)\sin 2\theta \\ \frac{1}{2}(\lambda - \mu)\sin 2\theta & \lambda\cos^2\theta + \mu\sin^2\theta \end{pmatrix}$$

We choose the solution

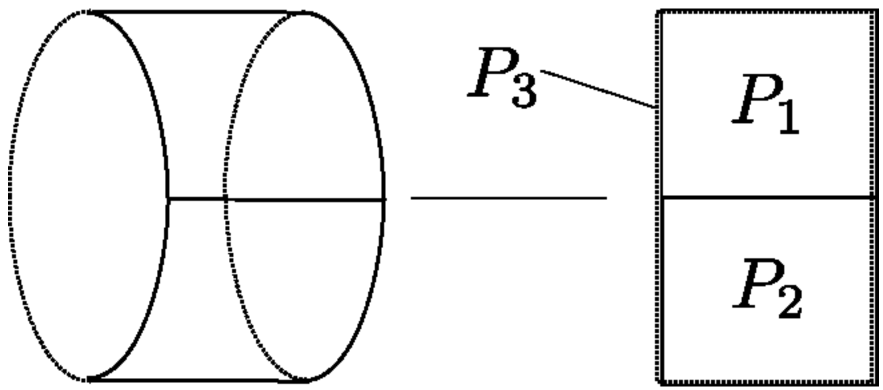
$$
\begin{aligned}
u|_{P_1 \cup P_3} &= \frac{1}{\lambda + \mu + (\mu - \lambda)\sin 2\theta} \\
&\times \begin{pmatrix} 1 & \frac{1}{2}(\lambda - \mu)\sin 2\theta - \mu\cos^2\theta - \lambda\sin^2\theta \\ 1 & \lambda\cos^2\theta + \mu\sin^2\theta + \frac{1}{2}(\mu - \lambda)\sin 2\theta \end{pmatrix} \begin{pmatrix} \omega_{1/10} \\ \rho_{3/4} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
u|_{P_2} &= \frac{1}{\lambda + \mu + (\lambda - \mu)\sin 2\theta} \\
&\times \begin{pmatrix} 1 & \frac{1}{2}(\mu - \lambda)\sin 2\theta - \lambda\cos^2\theta - \mu\sin^2\theta \\ 1 & \mu\cos^2\theta + \lambda\sin^2\theta + \frac{1}{2}(\lambda - \mu)\sin 2\theta \end{pmatrix} \begin{pmatrix} \omega_{1/10} \\ \rho_{3/4} \end{pmatrix}
\end{aligned}
$$

The matrices in the above formulas are essentially the inverses of the jump operators.

Experimental results:

Grid size	Max rel. error	Compr. time (secs./patch)	Sparse solve time
0.18	1E-3	2	0.001
0.15	5E-4	4	0.001
0.10	5E-6	45	0.001
0.09	2E-6	74	0.001
0.08	8E-7	115	0.001
0.06	2E-8	557	0.003



- Surface of cylinder on left is covered by 3 patches
- These patches are mapped bijectively onto 2 squares P_1, P_2 and a rectangle P_3
- P_3 exactly overlaps $P_1 \cup P_2$
- There are 2 left vertical edges in the boundary
- There are 2 right vertical edges in the boundary
- The top and bottom horizontal edges are not part of the boundary
- We chose the solution $w(\boldsymbol{x}) = \boldsymbol{x}_1^{5/2} \cos(5\pi \boldsymbol{x}_2/2)$

Experimental results:

Grid size	Max rel. error	Compr. time (secs./patch)	Sparse solve time
0.18	1E-2	4	0.002
0.10	8E-6	125	0.009
0.06	2E-6	1241	0.027

Note:

- Singular PDE
- Singular solution
- Non-trivial geometry

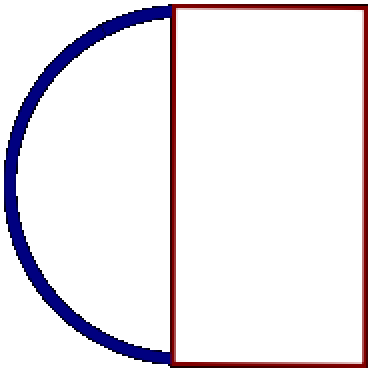
	Circle	Constant coefficient scalar elliptic

$$\nabla^T \nabla u - u = f$$

- Solution: $(1 + 10(x - y^2)^2)^{-1}$
- Domain: Circle of diameter 1
- Covered by two rectangular patches (no mapping required!)
- One-off code

Grid spacing	Error
0.1	2E-3
0.075	3E-4
0.05	4E-5
0.0375	1E-5
0.025	2E-6

	Half-circle plus rectangle	Constant coefficient div-curl

Domain: 

- Covered by 2 rectangular patches (no mapping required!)
- Solution

$$u = \begin{pmatrix} (1+x^2+y^2)^{-1} \\ x^2 - 2y^2 + x\,y - x + 1 \end{pmatrix}$$

- One-off code

Experimental results:

Grid spacing	Digits of accuracy
0.4	3
0.2	4
0.1	8

Summary

- Make the equations fat
- Choose a diagonal Sobolev norm
- Use high-relative accuracy numerical linear algebra techniques
- Convergence proof by compactness arguments
- Single Octave code $<$400 lines for all experiments, except curved geometries
- Code, papers, etc: `http://scg.ece.ucsb.edu/`

Future work

- Proper API for curved geometry yielding simple high-order solver
- Extension to inhomogenous jump conditions
- Applications to eigenvalue problems
- Applications to non-linear elliptic problems
- Extension to 3D

Thank you!