A minimum Sobolev norm numerical technique for PDEs

S. Chandrasekaran∗
ECE Dept.
Univ. of California, Santa Barbara

H. Mhaskar
Claremont Graduate School
Claremont

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A vector-valued function $u$ satisfies:

\[ D_1(x, u) = f(x), \quad x \in \Omega \]
\[ D_2(x, u) = g(x), \quad x \in \partial \Omega \]

- $D_i$ is a local linear differential operator with variable number of “rows”
- Find $u$ numerically
Current **Octave** code assumes following form of PDE

\[ A_1 \partial_1 u + A_2 \partial_2 u + Bu = f, \quad \text{on } \Omega \]
\[ Cu = g, \quad \text{on } \partial\Omega \]

where

\[
egin{align*}
  u & : \mathbb{R}^2 \to \mathbb{R}^q \\
  f & : \mathbb{R}^2 \to \mathbb{R}^p \\
  A_1, A_2, B & : \mathbb{R}^2 \to \mathbb{R}^{p \times q} \\
  r & : \mathbb{R}^2 \to \mathbb{N} \\
  C(x) & : \mathbb{R}^q \to \mathbb{R}^{r(x)}
\end{align*}
\]

- \( u \) has \( q \) components
- \( f \) has \( p \); not necessarily square
- Number of boundary conditions, \( r(x) \), is allowed to vary
- No assumptions of homogeneity
- First-order form
Our method also works with higher-order derivatives

FUD from previous attempts to use first-order form:

- Missing boundary conditions for extra variables in first-order form
- Mistaken assumption that discretized linear system must be square or skinny
- Large memory foot-print problem for first-order form
- Higher-order derivatives require more bits
- No known numerical work on variable coefficient fourth-order PDEs
- Seems to be missing from FEM, FD literature

Fat is a great alternative
- $\overline{\Omega}$ is covered by strictly convex quadrilaterals called patches
- Patches can overlap
- Curved boundaries don’t have to be approximated
On each patch we use modified 2D Chebyshev as basis

- $T_m(x) = \cos (m \cos^{-1} x)$ for $x \in [-1, 1]$
- $T_m(x) = T_{m_1}(x_1)T_{m_2}(x_2)$ for $\mathbb{N}^2$
- $\varphi_P$ be the homography from patch $P$ to $[-1, 1]^2$
- Bases on patch $P$: $T_m \circ \varphi_P$ for $m \in \mathbb{N}^2$
- Note that $\varphi_P$ is from a strictly convex quadrilateral to the cube even if the patch overlaps a curved boundary
- No mapping problem like that for curved finite elements
\[ u \mid_{\text{Patch}_1} = \sum_{m \in \mathbb{N}^2} \alpha_m T_m \circ \varphi_{\text{Patch}_1} \]
\[ u \mid_{\text{Patch}_2} = \sum_{m \in \mathbb{N}^2} \beta_m T_m \circ \varphi_{\text{Patch}_2} \]
We pick collocation as the discretization scheme.

Three types of grid points:

- Red points $x_i$ interior to each patch and open set $\Omega$.
- Green points $x_i$ on boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$.
- Blue points $x_i$ inside open set $\Omega$ and on interface edges shared between two patches.
- On each patch coefficients of Chebyshev expansions ($\alpha$ and $\beta$) are unknowns.

- On blue interface points on each edge common to two patches $u$ is an unknown.

![Diagram of patches and interface points](image)
- For each patch collocate PDE at red interior points

\[ \sum_{m \in \mathbb{N}^2} (A_1 \partial_1 + A_2 \partial_2 + B)(T_m \circ \varphi)(x_i) \alpha_m = f(x_i) \]

- For each patch collocate boundary condition at green boundary points

\[ \sum_{m \in \mathbb{N}^2} C(x_i)(T_m \circ \varphi)(x_i) \alpha_m = g(x_i) \]

- For each patch collocate continuity conditions at blue interface points

\[ \sum_{m \in \mathbb{N}^2} (T_m \circ \varphi)(x_i) \alpha_m = u(x_i) \]

- Note that \(u(x_i)\) are the only unknowns connecting equations across patches
The equations for the example problem:
Minimum Sobolev norm solution

- System is fat. Choose minimum norm solution. Which norm?

- Local $s$-Sobolev 2-norm on each patch

  $$\| u \|_{\text{Patch}_1}^2 \equiv \sum_{m \in \mathbb{N}^2} \| \alpha_m \|^2 (1 + \| m \|^2)^s = \| D_s \alpha \|^2_2$$

  where the standard Euclidean 2-norm uses

  $$D_s = \text{diag}((1 + \| m \|^2)^{s/2})$$

- Global $s$-Sobolev norm

  $$\| u \|_s^2 = \sum_{\text{Patch}} \| u \|_{\text{Patch}_1}^2$$

- Large $s$ leads to higher-order convergence. We use $s = 10$.

- Large $s$ leads to severely ill-conditioned systems. We use special solvers.
Standard solver

- Write the equation as

\[
\begin{pmatrix}
A_{11} & 0 & 0 \\
A_{21} & 0 & A_{23} \\
0 & A_{32} & 0 \\
0 & A_{42} & A_{43}
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
u_I
\end{pmatrix}
=
\begin{pmatrix}
fg \\
0 \\
fg \\
0
\end{pmatrix}
\]

- For minimum \(s\)-Sobolev 2-norm solution insert \(D_s\)

\[
\begin{pmatrix}
A_{11}D_s^{-1} & 0 & 0 \\
A_{21}D_s^{-1} & 0 & A_{23} \\
0 & A_{32}D_s^{-1} & 0 \\
0 & A_{42}D_s^{-1} & A_{43}
\end{pmatrix}
\begin{pmatrix}
D_s \alpha \\
D_s \beta \\
u_I
\end{pmatrix}
=
\begin{pmatrix}
fg \\
0 \\
fg \\
0
\end{pmatrix}
\]

- Compute ordinary minimum 2-norm solution using standard sparse \(LQ\) factorization.

- Convergence of solution (Golomb-Weinberger) can be established by standard compactness arguments using a variant of the Ascoli-Arzelia theorem with interpolation conditions.

- Assumptions include: existence & uniqueness of solution in appropriate Sobolev space, and linear independence of collocated equations.
Special solver

- For large $s$ values standard solver fails numerically
- Similar problem for classical high-order methods
- Our problem has the form well-conditioned fat matrix times highly ill-conditioned diagonal matrix
- Matrix was made fat to make it well-conditioned (similar to compressive sensing)
- For such under-determined problems special work by [Stewart], [Hough & Vavasis], [Gu], [Castro-Gonzalez, Ceballos, Dopico & Molera], [Higham], etc.
- Special two-sided orthogonal decomposition with complete pivoting
- Extension by us to sparse case; also greatly reduces memory consumption
- Used in all numerical experiments
- Truncation of expansion requires sophisticated analysis [Chandrasekaran & Mhaskar, JCP, 2013]
• Large domain \( \subseteq [0, 36] \times [0, 14] \)

• Outer boundary is not rectangle; includes wheels

• Covered by 45 patches

• \( p \)-convergence; so no refinement of mesh in these experiments
\[ \begin{align*}
\theta(x) &= \frac{x_1}{1 + x_2} \\
\lambda(x) &= \frac{1 + x_2}{1 + x_1} \\
\mu(x) &= \frac{1 + x_1}{1 + x_2} \\
A(\cdot) &= \begin{pmatrix}
\lambda \cos^2 \theta + \mu \sin^2 \theta & \frac{1}{2} (\mu - \lambda) \sin 2\theta \\
\frac{1}{2} (\mu - \lambda) \sin 2\theta & \mu \cos^2 \theta + \lambda \sin^2 \theta
\end{pmatrix} \\
\omega_a(x) &= \frac{1}{1 + a (x_1 - x_2^2)^2} \\
\rho_b(x) &= (1 + \|x\|_2^2)^b
\end{align*} \]

- \( A(x) > 0 \) whenever \( x > 0 \)
- \( A \) has variable eigenvalues and variable eigenvectors
- \( \omega_a \) has singularities on a parabola in \( \mathbb{C}^2 \) whose distance to the real plane \( \mathbb{R}^2 \) is controlled by \( a \)
- \( \rho_b \) is not a polynomial or a rational for \( b \notin \mathbb{Z} \)
Coefficients of PDE in first-order form

\[ A_1 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 1 \end{pmatrix} \]

\[ A_2 = \begin{pmatrix} A_{21} & A_{22} \\ -1 & 0 \end{pmatrix} \]

\[ B = \begin{pmatrix} \mu \cos \theta - \lambda \sin \theta & \lambda \cos \theta + \mu \sin \theta \\ 0 & 0 \end{pmatrix} \]

\[ u = \begin{pmatrix} \omega_1 \\ \rho_{1/4} \end{pmatrix} \]

\[ C = \tau^T \]

Known solution

Tangential boundary conditions on outer rectangle

Normal boundary conditions on car body

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<td>0.24</td>
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<td>6.3</td>
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</table>
\[ \nabla^T A \nabla v + b^T A \nabla v + cv = f_1 \]

Coefficients in $3 \times 3$ first-order form:

\[
A_1 = \begin{pmatrix} 0 & 1 & 0 \\ A_{11} & 0 & 0 \\ A_{21} & 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ A_{12} & 0 & 0 \\ A_{22} & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} c & b_1 & b_2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

\[ b = \begin{pmatrix} \mu \cos \theta - \lambda \sin \theta \\ \lambda \cos \theta + \mu \sin \theta \end{pmatrix} \quad c = -\sqrt{\lambda^2 + \mu^2} \]

\[ f = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \quad u = \begin{pmatrix} v \\ A \nabla v \end{pmatrix} \quad v = \omega_{1/10} \]

\[ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \nu_1 & \nu_2 \\ 0 & \tau_1 & \tau_2 \end{pmatrix} \text{ or } \begin{pmatrix} * & * & * \end{pmatrix} \]

Dirichlet \quad Neumann \quad Tangential \quad Mixed
Experimental results:

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<tr>
<td>0.24</td>
<td>8E-5</td>
<td>3192</td>
<td>39.5</td>
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- This includes error in (some linear combination of) derivatives of the solution
$E > 0$ is Young’s modulus, $-1 < v < \frac{1}{2}$ is Poisson’s ratio,

\[
\mathcal{D} = \frac{E}{(1+v)(1-2v)} \begin{pmatrix}
1 - v & v & 0 \\
v & 1 - v & 0 \\
0 & 0 & \frac{1}{2}(1 - 2v)
\end{pmatrix}, \quad E = \lambda, \quad v = \frac{\mu - 2\lambda}{2(\mu + \lambda)}
\]

$w$ is displacement, $\sigma$ is elastic stress tensor, $u$ is unknown,

\[
\sigma = \mathcal{D} \begin{pmatrix}
\frac{\partial}{\partial_1} & 0 \\
0 & \frac{\partial}{\partial_2}
\end{pmatrix} w, \quad w = \begin{pmatrix}
\omega^{1/10} \\
\rho^{3/4}
\end{pmatrix}, \quad u = \begin{pmatrix}
w \\
\sigma
\end{pmatrix} \in \mathbb{R}^5
\]
First-order $5 \times 5$ form coefficients:

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \mathcal{D}_{11} & 0 & 0 & 0 & 0 \\ \mathcal{D}_{21} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{D}_{33} & 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \mathcal{D}_{12} & 0 & 0 & 0 \\ 0 & \mathcal{D}_{22} & 0 & 0 & 0 \\ \mathcal{D}_{33} & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$f = \begin{pmatrix} -F_1 \\ -F_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$F$ is body force

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

**Displacement** boundary condition

$$C = \begin{pmatrix} 0 & 0 & \nu_1 & 0 & \nu_2 \\ 0 & 0 & 0 & \nu_2 & \nu_1 \end{pmatrix}$$

**Traction** boundary condition
We chose displacement boundary conditions everywhere.

Experimental results:

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<tr>
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<tr>
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<tr>
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<tr>
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<td>960</td>
<td>30.1</td>
</tr>
</tbody>
</table>

- This includes error in (some linear combination of) derivatives of the displacement (the elastic stress tensor)
Linearized stationary Navier-Stokes for incompressible flow

\[ b \text{ is base flow, } w \text{ is deviation from base flow, } p \text{ is pressure, } v \text{ is viscosity coeff.} \]

\[ -\nabla p + v\nabla^T \nabla w + (b^T \nabla) w + (w^T \nabla) b = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \nabla^T w = f_3 \]

We chose

\[ b = \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \quad v = \frac{1}{10} \quad w = \begin{pmatrix} \omega^{1/10} \\ \rho^{3/4} \end{pmatrix} \quad p(x) = \sin(x_1 - x_2) \]

Unknowns for $7 \times 7$ first-order form:

\[ u = \begin{pmatrix} w \\ p \\ \nabla w_1 \\ \nabla w_2 \end{pmatrix} \]
Coefficients of $7 \times 7$ first-order form:

$$A_1 = \begin{pmatrix} b_1 & 0 & -1 & v & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 & 0 & v & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} b_2 & 0 & 0 & 0 & v & 0 & 0 \\ 0 & b_2 & -1 & 0 & 0 & 0 & v \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \partial_1 b_1 & \partial_2 b_1 & 0 & 0 & 0 & 0 & 0 \\ \partial_1 b_2 & \partial_2 b_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
\[ C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \] Flow boundary conditions

\[ C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \] Pressure boundary condition

- Specified pressure on left and right outer vertical edges
- Specified flow everywhere else on boundary
- Note different number of boundary conditions on different parts of boundary

Experimental results:

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<tr>
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</tr>
<tr>
<td>0.47</td>
<td>7E-6</td>
<td>416</td>
<td>33.3</td>
</tr>
</tbody>
</table>
Exterior of car

Variable coefficient scalar fourth-order elliptic PDE

$$\Box = \begin{pmatrix} \partial_1^2 \\ \partial_1 \partial_2 \\ \partial_2^2 \end{pmatrix}$$

$$\Box^T = \begin{pmatrix} \partial_1^2 & \partial_1 \partial_2 & \partial_2^2 \end{pmatrix}$$

- $\mathcal{B}: \mathbb{R}^2 \to \mathbb{R}^{3 \times 3}$ take values that are symmetric positive-definite matrices
- $\mathcal{C}: \mathbb{R}^2 \to \mathbb{R}^{3 \times 2}$ and $\mathcal{C} \circ \nabla = (\Sigma_j c_{1j} \partial_j \Sigma_j c_{2j} \partial_j \Sigma_j c_{3j} \partial_j)$

PDE:

$$\Box^T \mathcal{B} \Box w + (\mathcal{C} \circ \nabla) \mathcal{B} \Box w + d^T \mathcal{B} \Box w + e^T \nabla w + cw = f_1$$

**Bi-harmonic** equation is a special case.

We chose

$$\mathcal{B} = \begin{pmatrix} 1 & \mu & 0 \\ \mu & 1 + \mu^2 & \lambda \\ 0 & \lambda & 1 + \lambda^2 \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \\ 1 & 1 \end{pmatrix} \quad d = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \mu \quad e = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \lambda$$

$$c = \rho^{3/4} \quad w = \omega^{1/100}$$
Unkowns for $9 \times 9$ first-order form:

\[
u = \begin{pmatrix}
    w \\
    \nabla w \\
    \mathcal{B} \Box w
\end{pmatrix}
\in \mathbb{R}^9
\]

Dirichlet and Neumann boundary conditions everywhere

\[
\mathcal{C} = \begin{pmatrix}
    1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & \nu_1 & \nu_2 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Coefficients of $9 \times 9$ first-order form:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & C_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{11} & B_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{23} & B_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{31} & B_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & C_{12} & C_{22} & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$
\[ B = \begin{pmatrix}
  c & e_1 & e_2 & d_1 & d_2 & d_3 & C_{11} & C_{21} & C_{32} \\
 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

Experimental results:

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- This includes error in (some linear combination of) third derivatives
Exterior of car  Poisson’s equation in polar coordinates

\[ x_1^2 \partial_1^2 w + x_1 \partial_1 w + \partial_2 w = f_1 \]

Coefficients of $3 \times 3$ first-order form:

\[
\begin{align*}
A_1 &= \begin{pmatrix}
x_1 & x_1^2 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} & A_2 &= \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix} & B &= \begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{pmatrix} & f &= \begin{pmatrix}
f_1 \\
0 \\
0 \\
\end{pmatrix}
\end{align*}
\]

with solution

\[
u = \begin{pmatrix}
w \\
\nabla w \\
\end{pmatrix}
\]

\[ w = x_1^{5/2} \omega_1(x) \]

Dirichlet boundary conditions everywhere \( C = (1\ 0\ 0) \)

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<td>10.3</td>
</tr>
<tr>
<td>0.30</td>
<td>2E-3</td>
<td>577</td>
<td>13.4</td>
</tr>
</tbody>
</table>
• Vertical axis is cable
• Horizontal axis is time
• Along cable
  – $V$ is voltage (unknown)
  – $I$ is current (unknown)
  – $C$ is capacitance
  – $L$ is inductance
  – $R$ is resistance
  – $G$ is conductance
• Telegraphers equation in $2 \times 2$ first-order form is hyperbolic

$$A_1 = \begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & R \\ G & 0 \end{pmatrix}, \quad u = \begin{pmatrix} V \\ I \end{pmatrix}$$

• Rather than $V(0, \ x_2)$ and $I(0, \ x_2)$ as initial conditions we provide $V(0, \ x_2)$ and $I(0, 36)$ as 2-point boundary conditions. Also $V$ is provided at cable ends.
• Cable geometry and topology changes with time (ill-posed?)
We chose space and time-varying cable parameters

\[ C = \lambda \quad L = \mu \quad R = \frac{\lambda}{2} + \mu \quad G = \lambda + \frac{\mu}{2} \quad V = \omega_{1/10} \quad I = \rho_{3/4} \]

Experimental results:

<table>
<thead>
<tr>
<th>Grid size</th>
<th>Max rel. error</th>
<th>Compr. time (secs./patch)</th>
<th>Sparse solve time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.14</td>
<td>9E-5</td>
<td>4</td>
<td>0.1</td>
</tr>
<tr>
<td>0.89</td>
<td>7E-5</td>
<td>13</td>
<td>0.1</td>
</tr>
<tr>
<td>0.73</td>
<td>3E-5</td>
<td>36</td>
<td>0.2</td>
</tr>
<tr>
<td>0.62</td>
<td>2E-5</td>
<td>92</td>
<td>0.3</td>
</tr>
<tr>
<td>0.53</td>
<td>5E-5</td>
<td>201</td>
<td>0.5</td>
</tr>
</tbody>
</table>

- Last row shows a stall
- We used much longer Chebyshev expansions in this test than the other ones
- We conjecture that an even longer expansion will get out of the stall, or, the problem is ill-posed
- 6 patches
- Thick line in the middle is a slit at $[-1, 1]$
- Outer rectangle is $[-2, 2] \times [-1, 1]$
Standard constant coefficient div-curl:

\[
A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}
\]

- \( \iota^2 = -1 \), \( z = x_1 + \iota x_2 \)

- \( (z^2 - 1)^{5/2} = u_R(x) + \iota u_I(x) \) with branch cut on \([-1, 1]\) which is also the slit in the rectangle

- \( u_R \) is continuous across slit

- \( u_I \) is dis-continuous across slit

- We choose solution as

\[
u = \begin{pmatrix} u_I \\ u_R \end{pmatrix}
\]
- Tangential boundary condition on outer boundary
- Single normal boundary condition on slit

Experimental results:

<table>
<thead>
<tr>
<th>Grid size</th>
<th>Max rel. error</th>
<th>Compr. time (secs./patch)</th>
<th>Sparse solve time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.29</td>
<td>6E-4</td>
<td>0.5</td>
<td>0.001</td>
</tr>
<tr>
<td>0.22</td>
<td>3E-4</td>
<td>1</td>
<td>0.001</td>
</tr>
<tr>
<td>0.18</td>
<td>2E-4</td>
<td>2</td>
<td>0.002</td>
</tr>
<tr>
<td>0.15</td>
<td>9E-5</td>
<td>4</td>
<td>0.002</td>
</tr>
<tr>
<td>0.09</td>
<td>1E-5</td>
<td>70</td>
<td>0.012</td>
</tr>
<tr>
<td>0.06</td>
<td>3E-6</td>
<td>533</td>
<td>0.035</td>
</tr>
</tbody>
</table>
- Normal boundary condition on outer boundary

- Double tangential boundary condition on slit; one as we approach slit from top, and one as we approach from bottom

<table>
<thead>
<tr>
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<th>Compr. time (secs./patch)</th>
<th>Sparse solve time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.29</td>
<td>1E-3</td>
<td>0.5</td>
<td>0.001</td>
</tr>
<tr>
<td>0.22</td>
<td>5E-4</td>
<td>1</td>
<td>0.001</td>
</tr>
<tr>
<td>0.18</td>
<td>2E-4</td>
<td>2</td>
<td>0.002</td>
</tr>
<tr>
<td>0.15</td>
<td>1E-4</td>
<td>4</td>
<td>0.003</td>
</tr>
<tr>
<td>0.09</td>
<td>1E-5</td>
<td>67</td>
<td>0.012</td>
</tr>
<tr>
<td>0.06</td>
<td>4E-6</td>
<td>523</td>
<td>0.037</td>
</tr>
</tbody>
</table>
• Contained in $[0, 3] \times [0, 2]$

• Covered by three patches $P_1, P_2, P_3$

• $P_2$ is a trapezoid; this is exploited in constructing solution

• Coefficient will be dis-continuous across edges of $P_2$

• Solution will satisfy a jump condition on those edges
• Coefficients of PDE in first-order form

\[
A_1 = \left( \begin{array}{cc} F_{11} & F_{12} \\ 0 & 1 \end{array} \right) \quad A_2 = \left( \begin{array}{cc} F_{21} & F_{22} \\ -1 & 0 \end{array} \right)
\]

\[
B = \left( \begin{array}{ccc} \mu \cos \theta - \lambda \sin \theta & \lambda \cos \theta + \mu \sin \theta \\ 0 & 0 \end{array} \right)
\]

• Jump condition at dis-continuity for this PDE

\[
\left( \begin{array}{c} \nu^T \mathcal{F}_+ \\ \tau^T \end{array} \right) u_+ = \left( \begin{array}{c} \nu^T \mathcal{F}_- \\ \tau^T \end{array} \right) u_-
\]

• \( \mathcal{F} \) makes a complicated jump across edges of \( P_2 \)

\[
\mathcal{F}\bigg|_{P_1 \cup P_3} = A \quad \mathcal{F}\bigg|_{P_2} = \left( \begin{array}{cc} \mu \cos^2 \theta + \lambda \sin^2 \theta & \frac{1}{2}(\lambda - \mu)\sin 2\theta \\ \frac{1}{2}(\lambda - \mu)\sin 2\theta & \lambda \cos^2 \theta + \mu \sin^2 \theta \end{array} \right)
\]
We choose the solution

\[ u|_{P_1 \cup P_3} = \frac{1}{\lambda + \mu + (\mu - \lambda) \sin 2\theta} \]
\[ \times \begin{pmatrix} 1 & \frac{1}{2}(\lambda - \mu) \sin 2\theta - \mu \cos^2 \theta - \lambda \sin^2 \theta \\ 1 & \lambda \cos^2 \theta + \mu \sin^2 \theta + \frac{1}{2}(\mu - \lambda) \sin 2\theta \end{pmatrix} \begin{pmatrix} \omega_{1/10} \\ \rho_{3/4} \end{pmatrix} \]

\[ u|_{P_2} = \frac{1}{\lambda + \mu + (\lambda - \mu) \sin 2\theta} \]
\[ \times \begin{pmatrix} 1 & \frac{1}{2}(\mu - \lambda) \sin 2\theta - \lambda \cos^2 \theta - \mu \sin^2 \theta \\ 1 & \mu \cos^2 \theta + \lambda \sin^2 \theta + \frac{1}{2}(\lambda - \mu) \sin 2\theta \end{pmatrix} \begin{pmatrix} \omega_{1/10} \\ \rho_{3/4} \end{pmatrix} \]

The matrices in the above formulas are essentially the inverses of the jump operators.
Experimental results:

<table>
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<tr>
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<th>Sparse solve time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.18</td>
<td>1E-3</td>
<td>2</td>
<td>0.001</td>
</tr>
<tr>
<td>0.15</td>
<td>5E-4</td>
<td>4</td>
<td>0.001</td>
</tr>
<tr>
<td>0.10</td>
<td>5E-6</td>
<td>45</td>
<td>0.001</td>
</tr>
<tr>
<td>0.09</td>
<td>2E-6</td>
<td>74</td>
<td>0.001</td>
</tr>
<tr>
<td>0.08</td>
<td>8E-7</td>
<td>115</td>
<td>0.001</td>
</tr>
<tr>
<td>0.06</td>
<td>2E-8</td>
<td>557</td>
<td>0.003</td>
</tr>
</tbody>
</table>
- Surface of cylinder on left is covered by 3 patches

- These patches are mapped bijectively onto 2 squares $P_1$, $P_2$ and a rectangle $P_3$

- $P_3$ exactly overlaps $P_1 \cup P_2$

- There are 2 left vertical edges in the boundary

- There are 2 right vertical edges in the boundary

- The top and bottom horizontal edges are not part of the boundary

- We chose the solution $w(\mathbf{x}) = \mathbf{x}_1^{5/2} \cos(5\pi \mathbf{x}_2/2)$
Experimental results:

<table>
<thead>
<tr>
<th>Grid size</th>
<th>Max rel. error</th>
<th>Compr. time (secs./patch)</th>
<th>Sparse solve time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.18</td>
<td>1E-2</td>
<td>4</td>
<td>0.002</td>
</tr>
<tr>
<td>0.10</td>
<td>8E-6</td>
<td>125</td>
<td>0.009</td>
</tr>
<tr>
<td>0.06</td>
<td>2E-6</td>
<td>1241</td>
<td>0.027</td>
</tr>
</tbody>
</table>

Note:

- Singular PDE
- Singular solution
- Non-trivial geometry
\[ \nabla^T \nabla u - u = f \]

- Solution: \((1 + 10(x - y^2)^2)^{-1}\)
- Domain: Circle of diameter 1
- Covered by two rectangular patches (no mapping required!)
- One-off code

<table>
<thead>
<tr>
<th>Grid spacing</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2E-3</td>
</tr>
<tr>
<td>0.075</td>
<td>3E-4</td>
</tr>
<tr>
<td>0.05</td>
<td>4E-5</td>
</tr>
<tr>
<td>0.0375</td>
<td>1E-5</td>
</tr>
<tr>
<td>0.025</td>
<td>2E-6</td>
</tr>
</tbody>
</table>
Half-circle plus rectangle  Constant coefficient div-curl

Domain:
- Covered by 2 rectangular patches (no mapping required!)
- Solution
  \[ u = \begin{pmatrix} (1 + x^2 + y^2)^{-1} \\ x^2 - 2y^2 + xy - x + 1 \end{pmatrix} \]
- One-off code

Experimental results:

<table>
<thead>
<tr>
<th>Grid spacing</th>
<th>Digits of accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>3</td>
</tr>
<tr>
<td>0.2</td>
<td>4</td>
</tr>
<tr>
<td>0.1</td>
<td>8</td>
</tr>
</tbody>
</table>
Summary

- Make the equations fat
- Choose a diagonal Sobolev norm
- Use high-relative accuracy numerical linear algebra techniques
- Convergence proof by compactness arguments
- Single Octave code < 400 lines for all experiments, except curved geometries
- Code, papers, etc: http://scg.ece.ucsb.edu/

Future work

- Proper API for curved geometry yielding simple high-order solver
- Extension to inhomogenous jump conditions
- Applications to eigenvalue problems
- Applications to non-linear elliptic problems
- Extension to 3D

Thank you!