A construction of linear bounded interpolatory operators on the torus

S. Chandrasekaran* H. N. Mhaskar†

Abstract

Let $q \ge 1$ be an integer. Given M samples of a smooth function of q variables, 2π–periodic in each variable, we consider the problem of constructing a q–variate trigonometric polynomial of spherical degree $\mathcal{O}(M^{1/q})$ which interpolates the given data, remains bounded (independent of M) on $[-\pi,\pi]^q$, and converges to the function at an optimal rate on the set where the data becomes dense. We prove that the solution of an appropriate optimization problem leads to such an interpolant. Numerical examples are given to demonstrate that this procedure overcomes the Runge phenomenon when interpolation at equidistant nodes on $[-1,1]$ is constructed, and also provides a respectable approximation for bivariate grid data, which does not become dense on the whole domain.

1 Introduction

Interpolation at equidistant nodes on the unit interval $[-1,1]$ is a very classical problem. In the first course in numerical analysis, one learns of the Newton divided difference algorithm to find such an interpolant, and the corresponding error formula. The Runge example, $x \mapsto (x^2+25)^{-1}$, shows that the sequence of these interpolants need not converge even if the target function is analytic on $[-1,1]$. In general, Faber's theorem [13, Theorem 2, p. 27] states that for any interpolation matrix on $[-1,1]$, there exists a continuous function on $[-1,1]$ such that the corresponding polynomials of interpolation to this function do not converge.

The situation changes drastically if one allows the degree of the interpolatory polynomial to be greater than the minimal required. Thus, the following Theorem 1.1 is a simple consequence of [17, Theorem 2.7, p. 52]. For the purpose of this exposition, we denote the class of all algebraic polynomials of degree at most m by Π_m, and define $\|f\|_{\infty,[-1,1]} := \sup_{t\in[-1,1]}|f(t)|$. We note that for n equidistant nodes on $[-1,1]$, the quantity d_n in the following theorem satisfies $d_n \ge 2/n$.

Theorem 1.1 *Let $x_{k,n} = \cos\theta_{k,n} \in [-1,1]$ be an arbitrary system of nodes ($0 \le \theta_{1,n} < \cdots < \theta_{n,n} \le \pi$) and let*

$$d_n := \min_{1\le k\le n-1}(\theta_{k+1,n} - \theta_{k,n}).$$

Then for any $\epsilon > 0$, there exist linear polynomial operators P_n on $C[-1,1]$ with the following properties: (a) *If $m = \lfloor \pi(1+\epsilon)/d_n \rfloor$ then $P_n(P) = P$ for all $P \in \Pi_m$,* (b) *for $f \in C[-1,1]$, $P_n(f) \in \Pi_N$ where $N = (\pi/d_n + 1)(1+3\epsilon)$,* (c) *$P(f, x_{k,n}) = f(x_{k,n})$ for $k = 1, \cdots, n$, and* (d)

$$\|f - P_n(f)\|_{\infty,[-1,1]} \le c \inf_{P\in\Pi_m} \|f - P\|_{\infty,[-1,1]}. \tag{1.1}$$

*Department of Electrical and Computer Engineering, University of California, Santa Barbara, Santa Barbara, CA 93106. The research of this author was supported, in part, by grants CCF-0515320 and CCF-0830604 from the NSF.

†Department of Mathematics, California State University, Los Angeles, California, 90032, U.S.A. The research of this author was supported, in part, by grant DMS-0605209 and its continuation DMS-0908037 from the National Science Foundation and grant W911NF-09-1-0465 from the U.S. Army Research Office.

In many engineering applications, one has to find a good approximation to an unknown *multivariate* target function which also interpolates the function at certain points, sometimes called landmarks. For example, in the problem of image registration, we are given a set of locations $x_j \in [-1,1]^2$ in the first image and a corresponding set of points $y_j \in [-1,1]^2$ in the second image. The idea is that the location x_j in the first image is the "same" as the location y_j in the second image. We then hope to find a map $g : [-1,1]^2 \to \mathbb{R}^2$ such that $g(x_j) = y_j$, and such that g satisfies some smoothness conditions. There are at least two reasons for insisting on interpolatory approximation in this situation. First, the locations might have been chosen at great costs, including human efforts. Second, if the registration is being done many times over a sequence of images (for example when we stitch together video frames to form a large image), then a non-interpolatory approximation will cause a drift between the first image and the last image in the sequence.

It is interesting to note that polynomial interpolation in multivariate setting has a totally different flavor than in the univariate setting; for example, even if one has exactly as many points as the dimension of the polynomial space involved, there might not exist an interpolant from that space. Even if an interpolant exists, the error bounds for approximation depend heavily on the geometry of the points. In [8], we proved that an analogue of Theorem 1.1 holds in practically any setting where the so called direct theorem of approximation holds, provided we drop the requirement of linearity. In particular, we proved analogous results in the multivariate setting. However, the results in [8] are not constructive, and do not yield linear operators.

The purpose of this paper is to develop algorithms to achieve near best polynomial approximations to smooth multivariate functions, which satisfy interpolatory constraints. Our constructions will work without requiring any specific locations for the points where the target function is evaluated. We refer to such data as *scattered data*. We do not require that the data become dense on the whole cube. In turn, our approximations may not converge on the whole cube. However, they will converge at the limit points of the data, and we will estimate the rate of convergence.

To motivate our construction, we revert to the univariate case of Theorem 1.1. We recall that there is a one to one correspondence between functions on $[-1,1]$ and even, 2π-periodic function on $\mathbb{R}$, given by $f^\circ(\theta) = f(\cos\theta)$. Moreover, $\|f^\circ\|_{\infty,[-\pi,\pi]} := \sup_{\theta\in[-\pi,\pi]} |f^\circ(\theta)| = \|f\|_{\infty,[-1,1]}$. Let $r \ge 1$ be an integer, and f° be r times continously differentiable on $[-\pi,\pi]$. In this discussion, we will write P_n in place of $P_n(f)$. In view of a theorem of Czipser and Freud [2], the estimate (1.1) implies that $\|P_n^{\circ(r)}\|_{\infty,[-\pi,\pi]} \le c\|f^{\circ(r)}\|_{\infty,[-\pi,\pi]}$. Therefore, the minimization problem "minimize $\|P^{\circ(r)}\|_{\infty,[-\pi,\pi]}$ over all $P \in \Pi_N$, subject to the constraints $P(x_{k,n}) = f(x_{k,n})$, $k = 1,\cdots,n$" has a solution P_n^* with the right bounds on the r-th derivative of its periodic version. In view of the Arzela–Ascoli theorem, this implies that any subsequence of the sequence $\{P_n^*\}$ has a uniformly convergent subsequence. If x_0 is a limit point of a subsequence $\{x_{k,n}\}_{n\in\Lambda}$, then it is not difficult to deduce using the interpolatory conditions that $\lim_{n\in\Lambda,n\to\infty} P_n^*(x_0) = f(x_0)$. In this paper, we will extend these ideas to the multivariate periodic setting. Instead of describing the smoothness of the functions in terms of derivatives, we will consider Sobolev classes. We will also consider minimization in arbitrary L^p norms; the L^1 norm being of recent interest from the point of view of compressed sensing. Some technical details, involving a construction of quasi–interpolatory polynomial operators, are required to prove the rate of convergence of our constructions. However, the bulk of the technical details is in the proof of the feasibility of the optimization problem. We will use Theorem 2.1 in [8] with the appropriate Sobolev spaces, and will need to prove the analogue of Theorem 3.2 in [8] also with approximation in Sobolev spaces rather than the space of continuous functions as in that theorem. Our main tool is the construction of a multivariate analogue of trigonometric polynomial frames constructed in [11, 12].

We state our main results Section 2, and illustrate them numerically in Section 3. The proofs of the results are given in Section 5, following some preparation of a technical nature in Section 4. At the first reading, it might help to skip this section, referring back to the various statements there on an as needed basis.

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2 Main results

In the sequel, $q \geq 1$ will denote a fixed integer, and we will think of 2π–periodic functions on $\mathbb{R}^q$ as functions on $[-\pi,\pi]^q$, tacitly identified with the q dimensional torus. Analogous to the univariate case, any function $f : [-1,1]^q \to \mathbb{R}$, corresponds uniquely to the 2π–periodic function f° on $\mathbb{R}^q$ by the correspondence

$$f^\circ(\theta_1, \cdots, \theta_q) = f(\cos\theta_1, \cdots, \cos\theta_q).$$

The symbol $\| \circ \|$ will denote the Euclidean norm of a vector in $\mathbb{R}^q$. Let $\mathbb{H}_n^q$ denote the class of all trigonometric polynomials in q variables with spherical order at most n; i.e.,

$$\mathbb{H}_n^q := \{ \sum_{\mathbf{k}\in\mathbb{Z}^s,\ \|\mathbf{k}\|\leq n} a_{\mathbf{k}} \exp(i\mathbf{k}\cdot(\circ)) \ : \ a_{\mathbf{k}} \in \mathbb{C}\}.$$

Here, we find it convenient to use the same notation even if n is not an integer. It is not difficult to see that multivariate algebraic polynomials on $[-1,1]^q$ correspond to the trigonometric polynomials of the same order which are symmetric in each of the variables. Therefore, in this paper, we are interested mainly in the interpolation of multivariate periodic functions; the results can also be applied trivially to the interpolation of functions on $[-1,1]^q$, with suitable smoothness conditions defined in terms of the corresponding periodic function.

If $1 \leq p \leq \infty$, $K \subset [-\pi,\pi]^q$ and $f : K \to \mathbb{C}$ are Lebesgue measurable, we write

$$\|f\|_{p,K} = \begin{cases} \left\{ \displaystyle\int_K |f(\mathbf{x})|^p d\mathbf{x} \right\}^{1/p}, & \text{if } 1 \leq p < \infty, \\ \displaystyle\operatorname*{ess\,sup}_{\mathbf{x}\in K} |f(\mathbf{x})|, & \text{if } p = \infty. \end{cases} \tag{2.1}$$

The symbol $L^p(K)$ denotes the class of all Lebesgue measurable functions f for which $\|f\|_{p,K} < \infty$, with the usual convention that two functions are considered equal if they are equal almost everywhere. If $K = [-\pi,\pi]^q$, we will omit its mention from the notations. If $1 < p < \infty$, we will write $p' := p/(p-1)$, and extend this notation to $p = 1, \infty$ by setting $1' = \infty$, $\infty' = 1$. If $f \in L^1$, the Fourier coefficients of f are defined by

$$\hat{f}(\mathbf{k}) := \frac{1}{(2\pi)^q} \int_{[-\pi,\pi]^q} f(\mathbf{x}) \exp(-i\mathbf{k}\cdot\mathbf{x}) d\mathbf{x}, \qquad \mathbf{k} \in \mathbb{Z}^q. \tag{2.2}$$

If $f \in L^p$, then its degree of approximation from $\mathbb{H}_n^q$ is defined by

$$E_{n,p}(f) := \inf_{T\in\mathbb{H}_n^q} \|f - T\|_p.$$

If $s \in \mathbb{R}$, $1 \leq p \leq \infty$, the Sobolev class W_s^p consists of all $f \in L^p$ for which there exists $f^{(s)} \in L^p$ such that

$$\widehat{f^{(s)}}(\mathbf{k}) = (\|\mathbf{k}\|^2 + 1)^{s/2} \hat{f}(\mathbf{k}), \qquad \mathbf{k} \in \mathbb{Z}^q.$$

We define

$$\|f\|_{W_s^p} := \|f^{(s)}\|_p, \tag{2.3}$$

and note that W_s^p is a Banach space. We observe that if Δ is the Laplacian operator on $\mathbb{R}^q$, and s is an even, positive integer, then $f^{(s)} = (\Delta + I)^{s/2} f$, where I is the indentity operator. In particular, in this case, the operator $f \mapsto f^{(s)}$ is a surface derivative operator on the torus identified with $[-\pi,\pi]^q$. An important property of the spaces W_s^p is given in the following proposition, which will be proved in Section 4.2. Here, and in the rest of this paper, the symbols $c, c_1, \cdots$ will denote generic positive constants, depending on such fixed parameters of the problem as p, s, q, etc. and other quantities explicitly indicated, but their value may different at different occurrences, even within a single formula. The notation $A \sim B$ means that $c_1 A \leq B \leq c_2 A$.

The following proposition, to be proved in Section 4.2, gives an integral representation of functions in W_s^p.

Proposition 2.1 *Let $1 \le p \le \infty$, $s > q/p$. Then there exists a function $K_s \in L^{p'}$ such that*

$$\widehat{K_s}(\mathbf{k}) = (\|\mathbf{k}\|^2 + 1)^{-s/2}, \qquad \mathbf{k} \in \mathbb{Z}^q. \tag{2.4}$$

If $f \in W_s^p$, then for almost all $\mathbf{x} \in [-\pi, \pi]^q$,

$$f(\mathbf{x}) = \frac{1}{(2\pi)^q} \int_{[-\pi,\pi]^q} K_s(\mathbf{x} - \mathbf{y}) f^{(s)}(\mathbf{y}) d\mathbf{y} = \frac{1}{(2\pi)^q} \int_{[-\pi,\pi]^q} K_s(\mathbf{y} - \mathbf{x}) f^{(s)}(\mathbf{y}) d\mathbf{y}. \tag{2.5}$$

In particular, f is almost everywhere equal to a continuous function. Denoting this continuous function again by f, we have for any $0 < s' < s - q/p$,

$$E_{2^n,\infty}(f) \le c 2^{-n(s-q/p)} \|f\|_{W_s^p}, \ \|f\|_\infty \le c \|f\|_{W_{s'}^\infty} \le c \|f\|_{W_s^p}. \tag{2.6}$$

We remark that in the case $p = 2$, one can take the following approach for interpolation of functions in W_s^2, $s > q/2$. The Golomb–Weinberger variation principle [3] can be used to show that the solution of the minimization problem

$$\text{minimize } \{\|g\|_{W_s^2} \ : \ g(\mathbf{y}_{j,n}) = f(\mathbf{y}_{j,n}), \ j = 1, \cdots, M_n\} \tag{2.7}$$

has a solution in the span of $K_{2s}(\circ - \mathbf{y}_{j,n})$, and therefore, can be found by solving an appropriate system of linear equations. The stability of this system as well as the error bounds can be estimated using known techniques from the theory of radial basis functions, for example, [14] (See Theorem 4.5 below). However, we are interested in finding polynomial interpolants for functions in W_s^p for $s > q/p$, without requiring $p = 2$.

As customary in the theory of interpolation, let Y be the interpolation matrix whose n-th row Y_n contains M_n vectors $\{\mathbf{y}_{j,n}\}_{j=1}^{M_n}$. Our theorems will depend upon two quantities, defined in (2.8) below, that measure the density of these points as well as their rareness. First, if $\mathcal{C} \subseteq [-\pi, \pi]^q$ and $\mathbf{x} \in [-\pi, \pi]^q$, we define

$$\operatorname{dist}(\mathcal{C}, \mathbf{x}) := \inf_{\mathbf{y} \in \mathcal{C}} \|\mathbf{x} - \mathbf{y}\|.$$

Further, if $K \subseteq [-\pi, \pi]^q$, we define the mesh norm $\delta(\mathcal{C}, K)$ (respectively, separation radius $\eta_\mathcal{C}$) of $\mathcal{C}$ by

$$\delta(\mathcal{C}, K) := \sup_{\mathbf{x} \in K} \operatorname{dist}(\mathcal{C}, \mathbf{x}), \ \eta_\mathcal{C} := (1/2) \inf_{\mathbf{x}, \mathbf{y} \in \mathcal{C}, \ \mathbf{x} \ne \mathbf{y}} \|\mathbf{x} - \mathbf{y}\|. \tag{2.8}$$

We will simplify our notation, and write $\delta_n(K)$ for $\delta(Y_n, K)$ and $\eta_n := \eta(Y_n)$.

Our first theorem, to be proved in Section 5, shows the feasibility of a procedure for finding interpolatory trigonometric polynomials in $\mathbb{H}_n^q$.

Theorem 2.1 *Let $1 \le p \le \infty$, $s > q/p$, Y be as above.*
(a) *There exists an integer N^* with $N^* \sim \eta_n^{-1}$ and a mapping $\mathbf{P} : W_s^p \to \mathbb{H}_{N^*}^q$ such that for every $f \in W_s^p$,*

$$\mathbf{P}(f, \mathbf{y}_{j,n}) = f(\mathbf{y}_{j,n}), \qquad j = 1, \cdots, M_n, \tag{2.9}$$

and

$$\|f - \mathbf{P}(f)\|_{W_s^p} \le c \inf\{\|f - T\|_{W_s^p} \ : \ T \in \mathbb{H}_{N^*}^q\}. \tag{2.10}$$

(b) *We consider the minimization problem*

$$\textit{minimize} \left\{ \frac{1}{N^{*q}} \sum_{0 \le \mathbf{k} \le 3N^*-1} |P^{(s)}(2\pi\mathbf{k}/(3N^*))|^p \right\}^{1/p}, \tag{2.11}$$

where the minimum is over all $P \in \mathbb{H}_{N^*}^q$, such that $P(\mathbf{y}_{j,n}) = f(\mathbf{y}_{j,n})$, $j = 1, \cdots, M_n$, and an appropriate interpretation is understood in the case $p = \infty$. There exists a solution of this problem, $\mathbb{P}_n^* = \mathbb{P}_n^*(p, Y_n, f) \in \mathbb{H}_{N^*}^q$, such that $\|\mathbb{P}_n^*\|_{W_s^p} \le c\|f\|_{W_s^p}$.

In practice, it seems that we can take $N^* = 4\eta_n^{-1}$ if $p = 2$ and $q = 1$. We note that the problem (2.11) has a unique solution if $p = 2$, and the corresponding operator P_n^* is linear in f.

The next theorem, to be proved in Section 5, examines the convergence properties of the sequence $\{\mathbb{P}_n^*\}$.

Theorem 2.2 *Let $1 \le p \le \infty$, $s > q/p$, $f \in W_s^p$, N^* and $\mathbb{P}_n^*$ be found as in Theorem 2.1.*
(a) *If Λ is a subsequence of positive integers, $\mathbf{x}_0 \in [\pi, \pi]^q$, and*

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} \mathrm{dist}(Y_n, \mathbf{x}_0) = 0, \tag{2.12}$$

then

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} \mathbb{P}_n^*(\mathbf{x}_0) = f(\mathbf{x}_0).$$

(b) *There exists a constant $\gamma = \gamma(p, q, s)$ (independent of n) with the following property. If $\mathbf{x}_0 \in [-\pi, \pi]^q$, and $\delta_n([\mathbf{x}_0 - \delta, \mathbf{x}_0 + \delta]^q) \le \gamma\delta$, then*

$$\|f - \mathbb{P}_n^*\|_{\infty,[\mathbf{x}_0-\delta,\mathbf{x}_0+\delta]^q} \le c\delta^{s-q/p}\|f\|_{W_s^p}. \tag{2.13}$$

The proof of Theorem 2.1 occupies a major part of this paper. We will use an abstract result from [8], quoted here as Lemma 5.1. To use this result, we need first to approximate carefully an arbitrary element of the span of $\{K_s(\circ - \mathbf{y}) : \mathbf{y} \in Y_{N^*}\}$ for a suitable value of N^* by trigonometric polynomials in $\mathbb{H}_{N^*}$; indeed, N^* will be determined so that this approximation works. In turn, this involves an estimation of the coefficients of this element in terms of the norms of this element, as well as a good approximation bound on K_s. In preparation, in Section 4.1, we introduce certain localized kenels and operators, and prove a number of technical results concerning these. These enable us after some further preparation to prove Proposition 2.1 and study some further properties of the kernel K_s in Section 4.2. The proof of Theorem 2.2(a), as expected, is a compactness argument. We also need to estimate the discrete norm used in (2.11) by the corresponding continuous norm. The necessary facts are stated in Lemmas 4.4 and 4.3. The proof of Theorem 2.2(b) is quite simple in the case when $q = 1$, $p = \infty$, and s in an integer: If there are s elements of Y_n in $K = [x_0 - \delta, x_0 + \delta]$, we take the Lagrange interpolatory polynomial L for f (and hence, $\mathbb{P}_{N^*}$) at these points. The elementary Newton error formula for interpolation yields

$$\|f - \mathbb{P}_{N^*}\|_{\infty,K} \le \|f - L\|_{\infty,K} + \|\mathbb{P}_{N^*} - L\|_{\infty,K} \le c\delta^s\|f^{(s)}\|_\infty.$$

The Newton formula does not hold in the multivariate case, and no similarly clean estimates are possible independently of the geometry of the points in question. Therefore, we use a result from [9], quoted as Proposition 4.5 below, to construct an analogue of L, which is not interpolatory, but utilizes only the values $f(\mathbf{y}_{j,n}) = \mathbb{P}_{N^*}(\mathbf{y}_{j,n})$. To take care of the technicalities of noninteger s and L^p norms other than $p = \infty$, we use the direct and converse theorems of approximation theory. Although these results are folklore, we could not find them in the literature in the form which we needed. Therefore, in Section 4.3, we review the results in the form in which we found them, and reconcile them to our needs. We also describe the construction of the algebraic polynomial approximation.

3 Numerical experiments

In this section, we will present numerical experiments that demonstrate the behavior of the method over a wide variety of situations, some of which do not satisfy the assumptions made in this paper. In the case when $p = 2$, the optimization problem (2.11) has a numerically effective closed-form solution. In this case, the problem is formulated easier directly in terms of the coefficients of the trigonometric polynomials:

$$\text{Find } \arg\min_{\{a_{\mathbf{k}}\}} \sum_{\|\mathbf{k}\| \le N^*} |a_{\mathbf{k}}|^2 (1 + \|\mathbf{k}\|^2)^s, \tag{3.1}$$

subject to the constraints

$$\sum_{\|\mathbf{k}\| \le N^*} a_{\mathbf{k}} \exp(i\mathbf{k} \cdot \mathbf{y}_{j,n}) = f(y_{j,n}), \qquad j = 1, \cdots, M_n.$$

Therefore, we assume in this section that $p = 2$, and refer the interpolant resulting as a solution of this problem as a *minimum Sobolev norm (MNS)* interpolant. In our computations below, we actually consider rectangular sums rather than the spherical sums as in (3.1). The term MNS interpolant will be used for all such minor variations.

We first consider the classical Runge phenomenon by interpolating the function $f(x) = (1 + 100x^2)^{-1}$ at equi-spaced points $\{-1 + 2j/(n+1)\}_{j=1}^{n}$ on the interval $[-1, 1]$. In Table 1 we show the results of our numerical experiments. The first column of the table shows the number of data points n used for interpolation. To avoid any special structure among the points, we chose the values of n as indicated, so as to be essentially (but not exactly) doubling from step to step. In all cases, we chose the order of the interpolatory polynomial to be $2n$, and computed the maximum error by sampling the MSN interpolant at $3n$ equi-spaced points. Columns 2–8 show the maximum interpolation error with different values of s. The maximum error decreases more rapidly with increasing s, but there are diminishing returns for higher values of s due to increasing condition numbers and the concomitant loss of numerical accuracy. To minimize this loss, we used a special algorithm that combines an LU factorization along with the traditional LQ factorization for solving the minimum norm problem [1]. Our computations show clearly that the interpolants converge; i.e., the Runge phenomenon has disappeared. For comparison we also show in the last column the approximation error from using a cubic-spline interpolant.

n	$s = 1.5$	$s = 2.5$	$s = 3.5$	$s = 4.5$	$s = 5.5$	$s = 6.5$	Spline
31	3.6212e-03	3.1000e-03	4.2114e-03	5.1510e-03	1.0535e-01	1.0127e+00	3.5710e-03
61	4.5758e-04	1.0681e-04	4.7994e-05	3.0439e-05	2.4317e-05	2.0644e-04	6.5167e-04
121	1.7844e-04	1.2509e-06	8.4610e-08	3.2494e-08	5.1699e-08	2.7949e-07	4.1035e-05
241	6.7863e-05	2.4084e-07	6.8299e-09	1.1385e-10	2.5882e-09	2.7632e-07	2.3897e-06
481	2.5210e-05	4.5175e-08	6.4371e-10	5.8196e-11	6.5261e-09	7.2588e-06	1.4739e-07
961	9.2203e-06	8.3199e-09	6.3179e-11	7.5275e-11	6.2423e-08	7.3983e-04	9.2473e-09

Table 1: Columns 2–7: maximum error for MSN interpolant of Runge's function $(1+100x^2)^{-1}$ on $[-1, 1]$ with n equi-spaced points and different choices of s. The order of the interpolant was $2n$ in all cases. Column 8: maximum error with cubic spline interpolant.

Next, we consider a two dimensional interpolation problem on a region inside the square $[-1, 1] \times [-1, 1]$. The target function is given in polar coordinates by

$$f(r, \theta) := |r - 1/4|^{1/8} |1 - r|^{4/5} \sin(r(2\cos\theta + \sin\theta)).$$

The function is singular on the circles of radii 0, 1/4 and 1. Furthermore the function does not satisfy the smoothness conditions of this paper. For the data points, we take those vertices of a square grid which lie in the indicated regions. If h is the length of each side of the squares in this grid, the target polynomial is a bivariate polynomial of coordinatewise degree $\lfloor 2/h \rfloor$. In the following tables, n denotes the number of grid points which lie in the region in question, and m is the dimension of the space of interpolatory polynomials.

In Table 2, we compute the maximum error of the interpolant in the annulus $1/2 < r < 3/4$ using approximately $4n$ grid points. The results are shown in Table 2. This particular annulus is well removed from the singularities of f. Therefore it is pleasing to see that the MSN interpolant approximates the underlying function very accurately. We report the maximum error in the annuli $3/4 < r < 19/20$ and $1/4 < r < 3/10$ for the same MSN interpolants in Tables 3 and 4 respectively. These annuli are significantly closer to the circles of radii $r = 1$ and $r = 1/4$ where the function is singular. Not surprisingly, the error is much larger here, but still usefully small.

Next, for the same function f, we restricted the samples to the region $\{\{r < 1/4\} \cup \{1 < r\}\} \cap \{[-1, 1] \times [-1, 1]\}$. Note that this region is essentially made up of 5 pieces. We used n equi-spaced

n	m	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
100	484	2.1979e-03	1.4012e-03	2.0732e-03	2.1384e-03	3.2363e-03
352	1764	9.0618e-04	1.0095e-03	1.5181e-03	2.7093e-03	5.0088e-03
1280	6724	2.4002e-04	1.4093e-04	2.0451e-04	2.1890e-04	3.4952e-04
4924	26244	2.3078e-04	1.0745e-05	2.3017e-05	3.4346e-05	1.4726e-04

Table 2: Maximum error of MSN interpolant in region $1/2 < r < 3/4$.

n	m	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
100	484	2.8778e-02	4.6506e-03	1.2650e-02	3.3905e-02	8.8795e-02
352	1764	3.5019e-03	6.8887e-04	5.4859e-03	2.5252e-02	7.2012e-02
1280	6724	1.4967e-03	3.3349e-04	7.7618e-04	6.3416e-03	2.9345e-02
4924	26244	1.4569e-04	1.8086e-04	4.1918e-04	9.2673e-04	3.2014e-02

Table 3: Maximum error of MSN interpolant in region $3/4 < r < 19/20$.

samples in the region and computed the MSN interpolant with m coefficients for different values of s. The maximum error in the region $r < 1/5$ is reported in Table 5, and in the region $1.1 < r$ in Table 6.

These experiments show that the proposed scheme can perform well even on difficult problems, especially in two dimensions where traditional interpolation schemes require much more work to achieve comparable accuracy. The proposed method requires special algorithms to execute efficiently, which will be discussed elsewhere. The ideas presented here can also be generalized to handle noisy and redundant observations. These matters will also be reported elsewhere [1].

4 Technical preparation

In this section, we present may technical results which are preparatory to the proof of the main results of Section 2. Our proof of Theorem 2.1 will require Theorem 4.2 and Theorem 4.4. Subsections 4.1 and 4.2 are devoted to the proof of these. In Subsection 4.1, we introduce a localized kernel and the corresponding operator which will be used throughout this paper, and prove a number of results regarding these. In particular, we use these results in Subsection 4.2 to prove Proposition 2.1 and establish a few other facts related to the kernel K_s. In Subsection 4.3, we review some well known properties of multivariate trigonometric and algebraic polynomial approximation, which will be used in the proof of Theorem 2.2.

4.1 Localized kernels

Let $q \ge 1$ be an integer. For $t > 0$, and $h : [0,\infty) \to \mathbb{R}$, we define formally

$$\Psi_t(h,\mathbf{x}) := \sum_{\mathbf{k}\in\mathbb{Z}^q} h(\|\mathbf{k}\|/t)\exp(i\mathbf{k}\cdot\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^q. \tag{4.1}$$

We set $\Psi_0(h,\mathbf{x}) := 1$ and $\Psi_t(h,\mathbf{x}) := 0$ if $t < 0$.

The following theorem summarizes the important localization estimate for the kernel Ψ_t, where we use the notation

$$\mathcal{D}f(u) = f'(u)/u.$$

Theorem 4.1 *Let $Q > (q+1)/2$ be an integer, $h : [0,\infty) \to [0,\infty)$ be a $Q-1$ times continuously differentiable function supported on $[0,1]$, with an absolutely continuous derivative $h^{(Q-1)}$. In addition, we assume that for some constants $0 < a < b < \infty$, $h(t) = 0$ if $t \ge b$, and $h'(t) = 0$ if $0 \le t \le a$. With $R = (q-1)/2 + Q$, we have*

$$|\Psi_t(h,\mathbf{x})| \le c\|\mathcal{D}^Q h\|_{1,[0,\infty)} \frac{t^q}{\min(1,(t\|\mathbf{x}\|)^R)}, \qquad \mathbf{x}\in[-\pi,\pi]^q,\ t > 0. \tag{4.2}$$

n	m	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
100	484	3.1104e-02	2.9877e-02	2.9187e-02	2.9011e-02	2.8334e-02
352	1764	6.1838e-02	5.2437e-02	4.8763e-02	4.6610e-02	4.5226e-02
1280	6724	1.2503e-02	8.0592e-03	7.8406e-03	8.6545e-03	9.2571e-03
4924	26244	2.0597e-02	1.4048e-02	1.0375e-02	8.6748e-03	8.6535e-03

Table 4: Maximum error of MSN interpolant in region $1/4 < r < 3/10$.

n	m	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
21	484	7.7266e-03	1.4501e-02	1.7311e-02	1.9202e-02	2.0064e-02
89	1764	2.1075e-03	1.8032e-03	2.0722e-03	2.1088e-03	2.1234e-03
401	6724	2.2433e-03	1.2778e-03	7.6602e-04	5.5885e-04	4.8267e-04
1637	26244	8.9254e-04	9.2024e-04	5.7556e-04	2.9398e-04	1.5738e-04

Table 5: Maximum error of MSN interpolant in region $r < 1/5$.

Further,

$$\max_{\mathbf{x}\in[-\pi,\pi]^q} |\Psi_t(h,\mathbf{x})| = \Psi_t(h,\mathbf{0}) \le ct^q \|\mathcal{D}^Q h\|_{1,[0,\infty)}, \qquad t>0, \tag{4.3}$$

and for $1 \le p \le \infty$,

$$\|\Psi_t(h,\circ)\|_p \le ct^{q/p'} \|\mathcal{D}^Q h\|_{1,[0,\infty)} \qquad t>0. \tag{4.4}$$

Here, the constants denoted by c may depend upon a, b, q, and Q only.

In order to prove this theorem, we recall that the Bessel function J_α can be defined for $\alpha > -1/2$, $t > 0$ by ([18, Formula (1.71.6)])

$$\begin{aligned} J_\alpha(t) &= \frac{(t/2)^\alpha}{\Gamma((2\alpha+1)/2)\Gamma(1/2)} \int_{-1}^1 e^{itu}(1-u^2)^{\alpha-1/2} du \\ &= \frac{(t/2)^\alpha}{\Gamma((2\alpha+1)/2)\Gamma(1/2)} \int_{-1}^1 e^{-itu}(1-u^2)^{\alpha-1/2} du. \end{aligned} \tag{4.5}$$

It is customary to define

$$J_{-1/2}(t) = \frac{(t/2)^{-1/2}}{\Gamma(1/2)} \cos t, \qquad t>0. \tag{4.6}$$

For $f \in L^1(\mathbb{R}^q)$, we define its inverse Fourier transform by

$$\tilde{f}(\mathbf{x}) = (2\pi)^{-q} \int_{\mathbb{R}^q} f(\mathbf{y}) \exp(i\mathbf{y}\cdot\mathbf{x}) d\mathbf{y}, \qquad \mathbf{x} \in \mathbb{R}^q. \tag{4.7}$$

Lemma 4.1 (a) *Let $h_0(\mathbf{x}) = h(\|\mathbf{x}\|)$, $\mathbf{x} \in \mathbb{R}^q$. Then*

$$\widetilde{h_0}(\mathbf{x}) = \frac{\|\mathbf{x}\|^{(2-q)/2}}{(2\pi)^{q/2}} \int_0^\infty h(s) J_{(q-2)/2}(s\|\mathbf{x}\|) s^{q/2} ds. \tag{4.8}$$

(b) *For $\alpha \ge 1/2$,*

$$\frac{d}{dt}(t^\alpha J_\alpha(t)) = t^\alpha J_{\alpha-1}(t). \tag{4.9}$$

(c) *We have*

$$|J_\alpha(t)| \le ct^{-1/2}, \qquad t>0. \tag{4.10}$$

n	m	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
21	484	1.9324e-01	2.6661e-01	5.9105e-01	1.2583e+00	1.5732e+00
89	1764	8.6889e-02	2.3262e-02	4.7827e-02	1.5319e-01	2.8550e-01
401	6724	4.4847e-02	1.3919e-02	9.1716e-02	3.9371e-01	6.6656e+00
1637	26244	1.6146e-02	1.5521e-02	1.4326e-01	3.0044e+00	7.2167e+01

Table 6: Maximum error of MSN interpolant in region $1.1 < r$.

PROOF. Part (a) is proved, except with a different notation in [15, Theorem 3.3, p. 155]. Part (b) is a straightforward consequence of the series expansion for J_α [18, Formula (1.71.1)]:

$$J_\alpha(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)}(t/2)^{2\alpha+2k}.$$

The estimate (4.10) follows from [18, Formula (7.31.5)]. □

PROOF OF THEOREM 4.1. Without loss of generality, we may assume in this proof that $\|\mathcal{D}^Q h\|_{1,[0,\infty)} = 1$. First, we prove (4.3). The first equation follows immediately from the definitions and the fact that $h(t) \ge 0$ for all t. Since $h(\|\mathbf{k}\|/t) = 0$ if $\|\mathbf{k}\| \ge bt$, $|h(u)| \le c$ for $u \in \mathbb{R}$, and the cardinality of the set $\{\mathbf{k} \in \mathbb{Z}^q : \|\mathbf{k}\| \le bt\}$ does not exceed $c_1 t^q$, we see from the definition that $0 \le \Psi_t(h, 0) \le c_2 t^q$. This proves the last inequality in (4.3).

In the proof of (4.2), we can assume that $t\|\mathbf{x}\| \ge 1$. In this proof only, let $h_0(\mathbf{x}) = h(\|\mathbf{x}\|)$, $\mathbf{x} \in \mathbb{R}^q$. In view of the Poisson summation formula [15, p. 251] (our notation is different), we have for $\mathbf{x} \in [-\pi, \pi]^q$,

$$\Psi_t(h, \mathbf{x}) = (2\pi)^q t^q \sum_{\mathbf{k}\in\mathbb{Z}^q} \widetilde{h}_0(t(\mathbf{x} + 2\pi\mathbf{k})). \tag{4.11}$$

Let $\mathbf{k} \in \mathbb{Z}^q$, $t(\mathbf{x} + 2\pi\mathbf{k}) = \mathbf{y}$, and $\|\mathbf{y}\| = r$. In view of Lemma 4.1(a), we have

$$\widetilde{h}_0(\mathbf{y}) = \frac{r^{1-q/2}}{(2\pi)^{q/2}} \int_0^\infty h(s) J_{(q-2)/2}(sr) s^{q/2} ds. \tag{4.12}$$

Let $\alpha \ge -1/2$. The equation (4.9) used with $\alpha + 1$ in place of α shows that

$$\int_0^u J_\alpha(rs) s^{\alpha+1} ds = r^{-\alpha-2} \int_0^{ru} J_\alpha(v) v^{\alpha+1} dv = r^{-\alpha-2}(ru)^{\alpha+1} J_{\alpha+1}(ru) = \frac{u^{\alpha+2} J_{\alpha+1}(u)}{ru}.$$

Consequently, an integration by parts in (4.12) yields that

$$\int_0^\infty h(s) J_\alpha(sr) s^{\alpha+1} ds = r^{-1} \int_0^\infty \mathcal{D}h(u) J_{\alpha+1}(ru) u^{\alpha+2} du.$$

Repeating this Q times, we obtain

$$\int_0^\infty h(s) J_\alpha(sr) s^{\alpha+1} ds = r^{-Q} \int_0^\infty \mathcal{D}^Q h(u) J_{\alpha+Q}(ru) u^{\alpha+Q+1} du.$$

We recall that $h'(u) = 0$ if $u \notin [a, b]$. Consequently,

$$\int_0^\infty h(s) J_\alpha(sr) s^{\alpha+1} ds = r^{-Q} \int_a^b \mathcal{D}^Q h(u) J_{\alpha+Q}(ru) u^{\alpha+Q+1} du.$$

In view of (4.10) and the fact that $\alpha + Q + 3/2 \ge 0$, we deduce that

$$\left| \int_0^\infty h(s) J_\alpha(sr) s^{\alpha+1} ds \right| \le c r^{-Q-1/2} \|\mathcal{D}^Q h\|_{1,[0,\infty)} = c r^{-Q-1/2}, \qquad \alpha \ge -1/2. \tag{4.13}$$

Using $q/2-1$ in place of α and substituting the resulting estimate into (4.12), we obtain that

$$\widetilde{h}_0(t(\mathbf{x}+2\pi\mathbf{k})) = \widetilde{h}_0(\mathbf{y}) \le cr^{1/2-q/2-Q}\|\mathcal{D}^Q h\|_{1,[0,\infty)} = \frac{c}{(t\|\mathbf{x}+2\pi\mathbf{k}\|)^R}. \tag{4.14}$$

When $\mathbf{k}\neq\mathbf{0}$, we have

$$\|\mathbf{x}+2\pi\mathbf{k}\| \ge |\mathbf{x}+2\pi\mathbf{k}|_\infty \ge 2\pi|\mathbf{k}|_\infty - \pi \ge \pi|\mathbf{k}|_\infty \ge \pi\|\mathbf{k}\|/\sqrt{q}.$$

Since $Q>(q+1)/2$, we have $R>q$, and hence,

$$\sum_{\mathbf{k}\in\mathbb{Z}^q,\ \mathbf{k}\neq\mathbf{0}} |\widetilde{h}_0(t(\mathbf{x}+2\pi\mathbf{k}))| \le ct^{-R}\sum_{\mathbf{k}\in\mathbb{Z}^q,\ \|\mathbf{k}\|\ge 1}\frac{1}{\|\mathbf{k}\|^R} \le c(t\|\mathbf{x}\|)^{-R}. \tag{4.15}$$

If $\mathbf{k}=\mathbf{0}$, (4.14) yields $|\widetilde{h}_0(t\mathbf{x})| \le c(t\|\mathbf{x}\|)^{-R}$. Together with (4.15) and (4.11), this implies (4.2).

Since $R>q$, we see from (4.2) that

$$\int_{\|\mathbf{x}\|\ge 1/t}|\Psi_t(h,\mathbf{x})|d\mathbf{x} \le ct^{q-R}\int_{\|\mathbf{x}\|\ge 1/t}\|\mathbf{x}\|^{-R}d\mathbf{x} = ct^{q-R}\int_{1/t}^\infty u^{q-1}u^{-R}du \le c.$$

Since (4.3) shows that $\int_{\|\mathbf{x}\|\le 1/t}|\Psi_t(h,\mathbf{x})|d\mathbf{x} \le c$ as well, we have proved (4.4) in the case when $p=1$. The estimate (4.4) in the case $p=\infty$ follows from (4.3). The general case is obtained using the convexity inequality

$$\|g\|_p \le \|g\|_\infty^{1/p'}\|g\|_1^{1/p}, \qquad g\in L^1\cap L^\infty,\ 1<p<\infty.$$

□

If $f\in L^1$, we define

$$\sigma_t(h,f,\mathbf{x}) := \frac{1}{(2\pi)^q}\int_{[-\pi,\pi]^q} f(\mathbf{y})\Psi_t(h,\mathbf{x}-\mathbf{y})d\mathbf{y}. \tag{4.16}$$

The following theorem summarizes some facts related to this operator.

Theorem 4.2 *Let h satisfy the conditions of Theorem 4.1, $1\le p\le\infty$, $f\in L^p$.*
(a) *We have*

$$\|\sigma_t(h,f)\|_p \le c\|\mathcal{D}^Q h\|_{1,[0,\infty)}\|f\|_p, \qquad t>0. \tag{4.17}$$

(b) *In particular, if $h(t)=1$ on $[0,1/2]$ and $h(t)=0$ on $[1,\infty)$, then*

$$E_{t,p}(f) \le \|f-\sigma_t(h,f)\|_p \le c\left(1+\|\mathcal{D}^Q h\|_{1,[0,\infty)}\right)E_{t/2,p}(f), \qquad t>0. \tag{4.18}$$

(c) *If $s>0$ and $f\in W_s^p$ then*

$$E_{t,p}(f) \le \frac{c_1}{t^s}\|\mathcal{D}^Q h\|_{1,[0,\infty)}E_{t,p}(f^{(s)}), \qquad t>0 \tag{4.19}$$

and

$$\|\sigma_t(h,f)^{(s)}\|_p \le c\|\mathcal{D}^Q h\|_{1,[0,\infty)}t^s\|\sigma_t(h,f)\|_p, \qquad t>0. \tag{4.20}$$

(d) (Bernstein inequality) *In particular, if $t\ge 0$ and $T\in\mathbb{H}_t^q$,*

$$\|T\|_{W_s^p} \le ct^s\|T\|_p. \tag{4.21}$$

Proof. In view of (4.4), $\|\Psi_t(h,\circ)\|_1 \le c\|\mathcal{D}^Q h\|_{1,[0,\infty)}$. The estimate (4.17) is now clear in the case $p=\infty$, and follows from Fubini's theorem in the case when $p=1$. An application of Riesz–Thorin theorem leads to the intermediate cases.

Next, let $h(t)=1$ on $[0,1/2]$ and $h(t)=0$ on $[1,\infty)$. Then $\sigma_t(h,T)=T$ for all $T\in\mathbb{H}_{t/2}^q$. Therefore,

$$\|f-\sigma_t(h,f)\|_p = \|f-T-\sigma_t(h,f-T)\|_p \le c\left(1+\|\mathcal{D}^Q h\|_{1,[0,\infty)}\right)\|f-T\|_p, \qquad T\in\mathbb{H}_{t/2}^q.$$

This implies (4.18).

Next, let $s \in \mathbb{R}$, and in this proof only, $g_t(u) = (h(u) - h(2u))/(u^2 + 1/t^2)^{s/2}$, $u \in [0, \infty)$, $t \in (0, \infty)$. Using the fact that $g_t(u) = 0$ if $u \in [0, 1/4]$, it is not difficult to verify that g_t satisfies the same conditions as h and $\|\mathcal{D}^Q g_t\|_{1,[0,\infty)} \le c\|\mathcal{D}^Q h\|_{1,[0,\infty)}$ with constant independent of t. Since $\sigma_t(h, f) - \sigma_{2t}(h, f) = t^{-s}\sigma_t(g_t, f^{(s)})$, this implies that

$$\|\sigma_t(h, f) - \sigma_{2t}(h, f)\|_p \le ct^{-s}\|f^{(s)}\|_p\|\mathcal{D}^Q h\|_{1,[0,\infty)}, \qquad t > 0,\ s \in \mathbb{R}. \tag{4.22}$$

Next, let $s > 0$. Then (4.22) leads to

$$\begin{aligned} E_{t,p}(f) &\le \|f - \sigma_t(h, f)\|_p = \left\|\sum_{k=0}^{\infty}(\sigma_{2^k t}(h, f) - \sigma_{2^{k+1}t}(h, f))\right\|_p \\ &\le \sum_{k=0}^{\infty}\|\sigma_{2^k t}(h, f) - \sigma_{2^{k+1}t}(h, f)\|_p \le ct^{-s}\|f^{(s)}\|_p\|\mathcal{D}^Q h\|_{1,[0,\infty)}, \end{aligned}$$

If $T \in \mathbb{H}_t$ satisfies $\|f^{(s)} - T^{(s)}\|_p \le 2E_{t,p}(f^{(s)})$, then this implies that

$$E_{t,p}(f) = E_{t,p}(f - T) \le ct^{-s}\|\mathcal{D}^Q h\|_{1,[0,\infty)}\|f^{(s)} - T^{(s)}\|_p \le ct^{-s}E_{t,p}(f^{(s)})\|\mathcal{D}^Q h\|_{1,[0,\infty)}.$$

This proves (4.19).

We observe that $\sigma_1(h, f) = \hat{f}(\mathbf{0}) = \sigma_1(h, f^{(s)})$. We observe also that (4.22) holds also for $s < 0$. So, if $s > 0$, we may apply (4.22) with $f^{(s)}$ in place of f and $-s$ in place of s to conclude that

$$\|\sigma_t(h, f^{(s)}) - \sigma_{2t}(h, f^{(s)})\|_p \le ct^s\|f\|_p\|\mathcal{D}^Q h\|_{1,[0,\infty)}.$$

Hence, for $n \ge 1$,

$$\begin{aligned} \|\sigma_{2^n}{}^{(s)}(h, f)\|_p &= \|\sigma_{2^n}(h, f^{(s)}\|_p = \left\|\sigma_1(h, f^{(s)}) + \sum_{k=0}^{n-1}\left(\sigma_{2^{k+1}}(h, f^{(s)}) - \sigma_{2^k}(h, f^{(s)})\right)\right\|_p \\ &\le \|\sigma_1(h, f^{(s)})\|_p + \sum_{k=0}^{n-1}\left\|\sigma_{2^{k+1}}(h, f^{(s)}) - \sigma_{2^k}(h, f^{(s)})\right\|_p \\ &\le c\|f\|_p\left\{1 + \sum_{k=0}^{n-1}2^{ks}\right\}\|\mathcal{D}^Q h\|_{1,[0,\infty)} \le c2^{ns}\|f\|_p\|\mathcal{D}^Q h\|_{1,[0,\infty)}. \end{aligned}$$

This leads to (4.20). The estimate (4.21) is obtained by using (4.20) with $2t$ in place of t and T in place of f, where we may use a fixed h, so that the constant is independent of the function h used in the rest of the statements of this theorem. □

Our next major goal is to prove Theorem 4.4. In this section, we develop the properties of the kernels Ψ_n which are required in this proof. Let $\{\mathbf{y}_j\}_{j=1}^M \subset [-\pi, \pi]^q$, $m \ge 1$ be an integer with

$$\min_{j \ne k}\|\mathbf{y}_j - \mathbf{y}_k\| \ge 1/m. \tag{4.23}$$

We note that this implies $M \le cm^q$. In the sequel, we will assume tacitly that $\{\mathbf{y}_j\}_{j=1}^M$ is one of the members of a sequence of finite subsets of $[-\pi, \pi]^q$. Thus, M and m are variables, and the constants are independent of these. If $\mathbf{a} = \{a_k\}_{k=0}^\infty$ is any sequence of complex numbers, we define

$$\|\mathbf{a}\|_{\ell^p} := \begin{cases} \left\{\sum_{k=0}^\infty |a_k|^p\right\}^{1/p}, & \text{if } 1 \le p < \infty, \\ \sup_{k \ge 0}|a_k|, & \text{if } p = \infty. \end{cases}$$

If $\mathbf{a}$ is in a Euclidean space $\mathbb{R}^D$, $\|\mathbf{a}\|_{\ell^p} := \|(0, a_1, \cdots, a_D, 0, \cdots)$.

Proposition 4.1 *Let $n \ge 1$ be an integer, $1 \le p \le \infty$, $\mathbf{a} \in \mathbb{R}^M$, h, Q, R be as in Theorem 4.1, and $G(\mathbf{x}) := \sum_{j=1}^M a_j \Psi_n(h, \mathbf{x} - \mathbf{y}_j)$, $\mathbf{x} \in [-\pi, \pi]^q$.*
(a) *We have*

$$\|G\|_p \le c n^{q/p'} \left\{1 + (m/n)^R\right\}^{1/p'} \|\mathcal{D}^Q h\|_{1,[0,\infty)} \|\mathbf{a}\|_{\ell^p}. \tag{4.24}$$

(b) *Suppose that there exists a compact interval $I \subset (0,1]$ and a constant $c_0 = c_0(h, I)$ such that $h(t) \ge c_0$ if $t \in I$. Then there exists $C_1 > 0$ depending on I, c_0, q, and Q such that for $n \ge C_1 m$,*

$$c_2 n^{-q/p'} \|G\|_p \le \|\mathcal{D}^Q h\|_{1,[0,\infty)} \|\mathbf{a}\|_{\ell^p} \le c_3 n^{-q/p'} \|G\|_p. \tag{4.25}$$

The proof requires a number of preparatory results, some of which we find of interest in their own right.

Proposition 4.2 *Let h satisfy the conditions of Theorem 4.1, $\{\mathbf{y}_j\}_{j=1}^M \subset [-\pi, \pi]^q$, $m \ge 1$ be an integer with $\min_{j \ne k} \|\mathbf{y}_j - \mathbf{y}_k\| \ge 1/m$. For integer $n \ge 1$ and $\mathbf{x} \in [-\pi, \pi]^q$,*

$$\sum_{j, \|\mathbf{x} - \mathbf{y}_j\| \ge 1/m} |\Psi_n(h, \mathbf{x} - \mathbf{y}_j)| \le c n^q (m/n)^R \|\mathcal{D}^Q h\|_{1,[0,\infty)}. \tag{4.26}$$

Hence,

$$\frac{1}{m^q} \sum_{j=1}^M |\Psi_n(h, \mathbf{x} - \mathbf{y}_j)| \le c (n/m)^q \left\{1 + (m/n)^R\right\} \|\mathcal{D}^Q h\|_{1,[0,\infty)}. \tag{4.27}$$

Proof. Without loss of generality, we may assume that $\|\mathcal{D}^Q h\|_{1,[0,\infty)} = 1$. In this proof only, let $\mathbb{Z}_k = \{j \ : \ k/m \le \|\mathbf{x} - \mathbf{y}_j\| \le (k+1)/m\}$, $k = 1, 2, \cdots$. We note that since the minimal separation amongst $\mathbf{y}_j$'s does not exceed $1/m$, there are at most $c k^{q-1}$ elements in each $\mathbb{Z}_k$. We note that since $Q > (q+1)/2$, $R = (q-1)/2 + Q > q$. In view of (4.2), we have

$$\begin{aligned}
\sum_{j, \|\mathbf{x} - \mathbf{y}_j\| \ge 1/m} |\Psi_n(h, \mathbf{x} - \mathbf{y}_j)| &\le c n^q \sum_{j, \|\mathbf{x} - \mathbf{y}_j\| \ge 1/m} (n \|\mathbf{x} - \mathbf{y}_j\|)^{-R} \\
&= c n^{q-R} \sum_{k=1}^\infty \sum_{j \in \mathbb{Z}_k} \|\mathbf{x} - \mathbf{y}_j\|^{-R} \le c n^{q-R} m^R \sum_{k=1}^\infty k^{q-1-R} \\
&\le c n^q (m/n)^R.
\end{aligned}$$

This proves (4.26).

In light of (4.23), the number of $\mathbf{y}_j$'s with $\|\mathbf{x} - \mathbf{y}_j\| \le 1/m$ is bounded independently of M and m. Hence, (4.3) implies that

$$\sum_{j, \|\mathbf{x} - \mathbf{y}_j\| \le 1/m} |\Psi_n(h, \mathbf{x} - \mathbf{y}_j)| \le c n^q.$$

Together with (4.26), this leads to (4.27). □

For $f : \{\mathbf{y}_j\} \to \mathbb{R}$, we will write

$$|||f|||_p = \begin{cases} \left\{\frac{1}{m^q} \sum_{j=1}^M |f(\mathbf{y}_j)|^p\right\}^{1/p}, & \text{if } 1 \le p < \infty, \\ \max_{1 \le j \le M} |f(\mathbf{y}_j)|, & \text{if } p = \infty. \end{cases}$$

Theorem 4.3 *Let $1 \le p \le \infty$. For any integer $n \ge 1$, and $T \in \mathbb{H}_n^q$, we have*

$$|||T|||_p \le c (n/m)^{q/p} \left\{1 + (m/n)^R\right\}^{1/p} \|T\|_p. \tag{4.28}$$

Proof. In this proof only, let $h : [0, \infty) \to [0, \infty)$ be a fixed, infinitely differentiable function, $h(t) = 1$ if $0 \le t \le 1/2$, $h(t) = 0$ if $t \ge 1$, and we choose $Q = q + 1$. The constants will depend upon this h, but

h being fixed in this proof, this dependence need not be specified. A comparison of Fourier coefficients shows that for $T \in \mathbb{H}^q_{2n}$,

$$T(\mathbf{y}) = \frac{1}{(2\pi)^q} \int_{[-\pi,\pi]^q} T(\mathbf{x}) \Psi_{4n}(h, \mathbf{x} - \mathbf{y}) d\mathbf{y}.$$

In view of (4.27), we obtain

$$\frac{1}{m^q} \sum_{j=1}^{M} |T(\mathbf{y}_j)| \le \|T\|_1 \max_{\mathbf{x} \in [-\pi,\pi]^q} \left\{ \frac{1}{m^q} \sum_{j=1}^{M} |\Psi_{4n}(h, \mathbf{x} - \mathbf{y}_j)| \right\} \le c(n/m)^q \left\{1 + (m/n)^R\right\} \|T\|_1.$$

If $f \in L^1$, we apply this estimate with $\sigma_{2n}(h, f)$ in place of T, and use Corollary 4.2 (with $p = 1$) to deduce that

$$|||\sigma_{2n}(h, f)|||_1 \le c(n/m)^q \left\{1 + (m/n)^R\right\} \|f\|_1.$$

In view of (4.17) (with $p = \infty$), it is clear that for $f \in L^\infty$,

$$|||\sigma_{2n}(h, f)|||_\infty \le \|\sigma_{2n}(h, f)\|_\infty \le c\|f\|_\infty.$$

An application of Riesz–Thorin interpolation theorem now implies that for $1 \le p \le \infty$, and $f \in L^p$,

$$|||\sigma_{2n}(h, f)|||_p \le c(n/m)^{q/p} \left\{1 + (m/n)^R\right\}^{1/p} \|f\|_p. \tag{4.29}$$

If $T \in \mathbb{H}^q_n$, then $\sigma_{2n}(h, T) = T$. Therefore, (4.29) implies (4.28). □

Proposition 4.3 below is perhaps well known. A proof can be found in [7, Proposition 6.1].

Proposition 4.3 *Let $M \ge 1$ be an integer, $\mathbf{A}$ be an $M \times M$ matrix whose (i, j)–th entry is $A_{i,j}$. $1 \le p \le \infty$, and $\alpha \in [0, 1)$. If*

$$\sum_{\substack{i=1\\i\neq j}}^{M} |A_{j,i}| \le \alpha|A_{j,j}|, \ \sum_{\substack{i=1\\i\neq j}}^{M} |A_{i,j}| \le \alpha|A_{j,j}|, \qquad j = 1, \cdots, M, \tag{4.30}$$

and $\lambda = \min_{1\le i\le M} |A_{i,i}| > 0$, then $\mathbf{A}$ is invertible, and

$$\|\mathbf{A}^{-1}\mathbf{b}\|_{\ell^p} \le ((1 - \alpha)\lambda)^{-1}\|\mathbf{b}\|_{\ell^p}, \qquad \mathbf{b} \in \mathbb{R}^M. \tag{4.31}$$

We are now in a position to prove Proposition 4.1.

Proof of Proposition 4.1. Without loss of generality, we may assume that $\|\mathcal{D}^Q h\|_{1,[0,\infty)} = 1$. In view of (4.27), we have for $\mathbf{x} \in [-\pi, \pi]^q$,

$$|G(\mathbf{x})| \le \sum_{j=1}^{M} |a_j||\Psi_n(h, \mathbf{x} - \mathbf{y}_j)| \le \|\mathbf{a}\|_{\ell^\infty} \sum_{j=1}^{M} |\Psi_n(h, \mathbf{x} - \mathbf{y}_j)| \le cn^q \left\{1 + (m/n)^R\right\} \|\mathbf{a}\|_{\ell^\infty}.$$

Thus,

$$\|G\|_\infty \le cn^q \left\{1 + (m/n)^R\right\} \|\mathbf{a}\|_{\ell^\infty}.$$

Using (4.4), we see that

$$\|G\|_1 \le \sum_{j=1}^{M} |a_j|\|\Psi_n(h, \circ - \mathbf{y}_j)\|_1 \le c\|\mathbf{a}\|_{\ell^1}.$$

An application of Riesz–Thorin interpolation theorem with the operator $\mathbf{a} \mapsto \sum_{j=1}^{M} a_j \Psi_n(h, \circ - \mathbf{y}_j)$ implies (4.24).

Next, if the hypothesis in part (b) is satified, then

$$\Psi_n(h, \mathbf{0}) \ge \sum_{\mathbf{k},\ \|\mathbf{k}\|/n \in I} h(\|\mathbf{k}\|/n) \ge cn^q. \tag{4.32}$$

Therefore, (4.23) and (4.26) show that for $n \ge C_1 m$, $\ell = 1, \cdots, M$,

$$\sum_{\substack{j=1\\ j\neq\ell}} |\Psi_n(h, \mathbf{y}_\ell - \mathbf{y}_j)| \le (1/2)\Psi_n(h, \mathbf{0}). \tag{4.33}$$

In this proof only, let $\mathbf{A}$ be the matrix whose (ℓ, j)-th entry is $\Psi_n(h, \mathbf{y}_\ell - \mathbf{y}_j)$ and $\mathbf{b} \in \mathbb{R}^M$ be defined by $b_\ell = G(\mathbf{y}_\ell)$, $\ell = 1, \cdots, M$. In view of (4.33), (4.30) is satisfied with 1/2 in place of α, and in view of (4.32), we may choose λ to be cn^q. Hence, Proposition 4.3 implies that $\mathbf{A}$ is invertible, and

$$\|\mathbf{A}^{-1}\mathbf{b}\|_{\ell^p} \le cn^{-q}\|\mathbf{b}\|_{\ell^p}.$$

Since, $\mathbf{A}^{-1}\mathbf{b} = \mathbf{a}$, we have proved that

$$\|\mathbf{a}\|_{\ell^p} \le cn^{-q}\|\mathbf{b}\|_{\ell^p}. \tag{4.34}$$

Since $G \in \mathbb{H}_n^q$, we obtain from Theorem 4.3 that

$$|||G|||_p = m^{-q/p}\|\mathbf{b}\|_{\ell^p} \le c(n/m)^{q/p}\left\{1 + (m/n)^R\right\}^{1/p}\|G\|_p.$$

Since $n \ge C_1 m$, this gives

$$\|\mathbf{b}\|_{\ell^p} \le cn^{q/p}\|G\|_p.$$

Together with (4.34), this leads to the second inequality in (4.25). The first inequality follows from (4.24) and the fact that $n \ge C_1 m$. $\square$

4.2 Sobolev kernel

Our goal in this section is to prove Proposition 2.1 and Theorem 4.4, and establish a few other facts regarding the kernel K_s. In particular, we will give in Theorem 4.5 an estimate for the norm of the interpolation matrix $K_{2s}(y_j - y_k)$. In the sequel, we assume $Q > (q+1)/2$ is an integer, $h : [0, \infty) \to [0, \infty)$ is a fixed, $Q - 1$ times continuously differentiable function with an absolutely continuous derivative $h^{(Q-1)}$, $h(t) = 1$ if $0 \le t \le 1/2$, $h(t) = 0$ if $t \ge 1$, and h is nondecreasing on $[0, \infty)$. We will write $g(t) = h(t) - h(2t)$. Since h is fixed, the dependence of various constants on h need not be indicated. For $s \in \mathbb{R}$, we will write

$$\tilde{\Psi}_{n,s}(\mathbf{x}) := \sum_{\mathbf{k}\in\mathbb{Z}^q} g(\|\mathbf{k}\|/2^n)(\|\mathbf{k}\|^2 + 1)^{-s/2}\exp(i\mathbf{k}\cdot\mathbf{x}). \tag{4.35}$$

The following lemma lists some interesting properties of $\tilde{\Psi}_{n,s}$.

Lemma 4.2 *Let $s \in \mathbb{R}$. We have*

$$|\tilde{\Psi}_{n,s}(\mathbf{x})| \le c\frac{2^{n(q-s)}}{\min(1, (2^n\|\mathbf{x}\|)^R)}, \qquad \mathbf{x} \in [-\pi, \pi]^q. \tag{4.36}$$

Further,

$$\max_{\mathbf{x}\in[-\pi,\pi]^q} |\tilde{\Psi}_{n,s}(\mathbf{x})| = \tilde{\Psi}_{n,s}(\mathbf{0}) \sim 2^{n(q-s)}, \tag{4.37}$$

and for $1 \le p \le \infty$,

$$\|\tilde{\Psi}_{n,s}\|_p \le c2^{n(q/p'-s)}. \tag{4.38}$$

Proof. In this proof only, let $g_n(t) = g(t)/(t^2 + 1/n^2)^{s/2}$. Then for $\mathbf{x} \in [-\pi, \pi]^q$,

$$\tilde{\Psi}_{n,s}(\mathbf{x}) = 2^{-ns}\Psi_{2^n}(g_{2^n}, \mathbf{x}). \tag{4.39}$$

Each g_n satisfies the conditions of Theorem 4.1, with $a = 1/4$, $b = 1$. Moreover, $\|\mathcal{D}^Q g_n\|_{1,[0,\infty)} \le c$. Therefore, all assertions of the lemma, except for the second relation in (4.37), follow directly from

Theorem 4.1. Theorem 4.1 also implies that $\tilde{\Psi}_{n,s}(\mathbf{0}) \le c2^{n(q-s)}$. Since $g(1/2) = h(1/2) - h(1) = 1$, and g is continuous, there exists a nondegenerate interval $I \subset [1/4, 1]$ such that $g(t) \ge 1/2$ if $t \in I$. Hence,

$$\tilde{\Psi}_{n,s}(\mathbf{0}) = \sum_{\mathbf{k}\in\mathbb{Z}^q} g(\|\mathbf{k}\|/2^n)(1+\|\mathbf{k}\|^2)^{-s/2} \ge \sum_{\mathbf{k}\in\mathbb{Z}^q,\ \|\mathbf{k}\|/2^n\in I} g(\|\mathbf{k}\|/2^n)(1+\|\mathbf{k}\|^2)^{-s/2} \ge c2^{n(q-s)}.$$

This proves the second relation in (4.37). □

Proof of Proposition 2.1. Since $s > q/p$, (4.38) used with p' in place of p shows that

$$\sum_{n=0}^{\infty} \|\tilde{\Psi}_{n,s}\|_{p'} \le c\sum_{n=0}^{\infty} 2^{n(q/p-s)} < \infty.$$

So, the sequence of trigonometric polynomials, defined by

$$P_N(\mathbf{x}) = 1 + \sum_{n=0}^{N} \tilde{\Psi}_{n,s}(\mathbf{x}) = 1 + \sum_{n=0}^{N}\sum_{\mathbf{k}\in\mathbb{Z}^q} g(\|\mathbf{k}\|/2^n)(1+\|\mathbf{k}\|^2)^{-s/2}\exp(i\mathbf{k}\cdot\mathbf{x})$$

converges in $L^{p'}$. All the sums in the above expression being finite sums, we obtain for $N \ge 0$,

$$P_N(\mathbf{x}) = 1 + \sum_{\mathbf{k}\in\mathbb{Z}^q}\sum_{n=0}^{N} g(\|\mathbf{k}\|/2^n)(1+\|\mathbf{k}\|^2)^{-s/2}\exp(i\mathbf{k}\cdot\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^q} h(\|\mathbf{k}\|/2^N)(1+\|\mathbf{k}\|^2)^{-s/2}\exp(i\mathbf{k}\cdot\mathbf{x}).$$

If $\mathbf{k} \in \mathbb{Z}^q$, and $2^N \ge 2\|\mathbf{k}\|$, then $h(\|\mathbf{k}\|/2^N) = 1$, and $\hat{P}_N(\mathbf{k}) = (1+\|\mathbf{k}\|^2)^{-s/2}$. Denoting the $L^{p'}$–limiting function of P_N by K_s, it follows that $K_s \in L^{p'}$ and satisfies (2.4). Moreover, $P_N = \sigma_{2^N}(h, K_s)$, and the bound on $\|\tilde{\Psi}_{n,s}\|_{p'}$ in (4.38) used with p' in place of p shows that

$$\|K_s - \sigma_{2^N}(h, K_s)\|_{p'} \le c2^{N(q/p-s)}, \qquad N = 0, 1, 2, \cdots. \tag{4.40}$$

Both sides of the first equation in (2.5) have the same Fourier coefficients, and hence, they are equal almost everywhere. Similarly, a comparison of Fourier coefficients shows that $K_s(-\mathbf{x}) = K_s(\mathbf{x})$ for almost all $\mathbf{x}$. This implies the second equation in (2.5).

For $f \in W_s^p$, a comparison of Fourier coefficients again shows that for integer $m \ge 0$,

$$\sigma_{2^m}(h, f, \mathbf{x}) = \frac{1}{(2\pi)^q}\int_{[-\pi,\pi]^q} \sigma_{2^m}(h, K_s, \mathbf{x}-\mathbf{y}) f^{(s)}(\mathbf{y})d\mathbf{y}.$$

So, (4.40) implies that

$$\|\sigma_{2^m}(h,f) - \sigma_{2^{m-1}}(h,f)\|_\infty \le \frac{1}{(2\pi)^q}\|\sigma_{2^m}(h,K_s) - \sigma_{2^{m-1}}(h,K_s)\|_{p'}\|f^{(s)}\|_p \le c2^{m(q/p-s)}\|f^{(s)}\|_p.$$

Hence, the series $\sigma_1(h,f) + \sum_{m=1}^{\infty}(\sigma_{2^m}(h,f) - \sigma_{2^{m-1}}(h,f))$ converges uniformly. It is clear that this limit is almost everywhere equal to f, and by choosing the continuous representer in the equivalence class of f to be f, the limit is f. Moreover,

$$E_{2^n,\infty}(f) \le \|f - \sigma_{2^n}(h,f)\|_\infty \le \sum_{m=n}^{\infty}\|\sigma_{2^m}(h,f) - \sigma_{2^{m-1}}(h,f)\|_\infty \le 2^{n(q/p-s)}\|f^{(s)}\|_p.$$

This implies the first estimate in (2.6) is now clear. The second set of estimates are proved similarly. □

Our proof of Theorem 2.1 requires the following theorem that describes an approximation of a typical element of the span of $\{K_s(\circ - \mathbf{y}_j)\}$. We recall that the solution of the minimization problem (2.7) is in this span (with $2s$ in place of s).

Theorem 4.4 *Let $1 \le p \le \infty$, $s > q/p$, $\{a_j\}_{j=1}^M \subset \mathbb{R}$, $G(\mathbf{x}) = \sum_{j=1}^M a_j K_s(\mathbf{x} - \mathbf{y}_j)$, $\mathbf{x} \in [-\pi,\pi]^q$, and $m \ge 1$ be the smallest integer such that $\min_{\mathbf{y}_j \ne \mathbf{y}_k}\|\mathbf{y}_j - \mathbf{y}_k\| \ge 1/m$. Then there exists an integer N^*, independent of G, such that $N^* \sim m$ and*

$$\|G - \sigma_{N^*}(h, G)\|_{p'} \le (1/2)\|G\|_{p'}. \tag{4.41}$$

PROOF. As in the proof of Lemma 4.2, in this proof only, we write $g_n(t) = g(t)/(t+1/n)^s$. Then each g_n satisfies the conditions of Theorem 4.1, with $a = 1/4$, $b = 1$. Moreover, $\|\mathcal{D}^Q g_n\|_{1,[0,\infty)} \sim 1$, and (4.39) holds. In this proof only, let

$$G_n(\mathbf{x}) := \sum_{j=1}^{M} a_j \tilde{\Psi}_{n,s}(\mathbf{x}-\mathbf{y}_j) = \sigma_{2^n}(h,G,\mathbf{x}) - \sigma_{2^{n-1}}(h,G,\mathbf{x}), \qquad \mathbf{x} \in [-\pi,\pi]^q.$$

Then (4.17) implies that $\|G_n\|_{p'} \le c\|G\|_{p'}$. Moreover, the proof of Proposition 2.1 shows that

$$G(\mathbf{x}) - \sigma_{2^N}(h,G,\mathbf{x}) = \sum_{n=N}^{\infty} G_n(\mathbf{x}), \tag{4.42}$$

with convergence in the sense of $L^{p'}$.

In view of (4.39), (4.25) applied with $\Psi_{2^n}(g_n)$ yields that for $n \ge \log_2(C_1 m)$,

$$c_2 2^{n(s-q/p)}\|G_n\|_{p'} \le \|\mathbf{a}\|_{\ell^{p'}} \le c2^{n(s-q/p)}\|G_n\|_{p'}. \tag{4.43}$$

We now choose L so that 2^L is the smallest power of 2 exceeding $C_1 m$. Then the second inequality in (4.43), used with L in place of n, gives

$$\|\mathbf{a}\|_{\ell^{p'}} \le cm^{(s-q/p)}\|G_L\|_{p'} \le cm^{(s-q/p)}\|G\|_{p'}. \tag{4.44}$$

From (4.42), (4.43), and (4.44), we conclude that for $2^N \ge C_1 m$,

$$\|G - \sigma_{2^N}(h,G)\|_{p'} \le \sum_{n=N}^{\infty} \|G_n\|_{p'} \le c\|\mathbf{a}\|_{\ell^{p'}} \sum_{n=N}^{\infty} 2^{-n(s-q/p)} \le c(m2^{-N})^{(s-q/p)}\|G\|_{p'}.$$

We now choose N so that $2^N \sim m$ and the last term above is at most $(1/2)\|G\|_{p'}$, and set $N^* = 2^N$. $\square$

We note a consequence of the proof, which might be of independent interest in view of the fact that the interpolant which yields the minimal Sobolev norm amongst all interpolants is in the span of $\{K_{2s}(\circ - \mathbf{y}_k)\}_{k=1}^M$. The following theorem gives the norm of the inverse of the interpolation matrix $(K_{2s}(y_j - y_k))$ in terms of the minimal separation (equivalently, m).

Theorem 4.5 *Let $s > q/2$, and $\mathcal{I}$ be the $M \times M$ matrix whose (j,k)-th entry is $K_{2s}(y_j - y_k)$, where the points $\mathbf{y}_j$ satisfy (4.23). Then $\mathcal{I}$ is positive definite, and*

$$\|\mathcal{I}^{-1}\| \le cm^{s-q/2}. \tag{4.45}$$

PROOF. We observe that a comparison of Fourier coefficients shows that

$$K_{2s}(\mathbf{y}_j - \mathbf{y}_k) = \frac{1}{(2\pi)^q}\int_{[-\pi,\pi]^q} K_s(\mathbf{x}-\mathbf{y}_j)K_s(\mathbf{x}-\mathbf{y}_k)d\mathbf{y}.$$

Let $\mathbf{a} \in \mathbb{R}^M$, and $G = \sum_{j=1}^M a_j K_s(\circ - \mathbf{y}_j)$. Then the above identity leads to

$$\sum_{k,j=1}^{M} a_j a_k K_{2s}(\mathbf{y}_j - \mathbf{y}_k) = \frac{1}{(2\pi)^q}\int_{[-\pi,\pi]^q} \left|\sum_{j=1}^{M} a_j K_s(\mathbf{x}-\mathbf{y}_j)\right|^2 d\mathbf{y} = \|G\|_2^2. \tag{4.46}$$

The estimate (4.44) used with $p = 2$ now shows that

$$\sum_{k,j=1}^{M} a_j a_k K_{2s}(\mathbf{y}_j - \mathbf{y}_k) \ge cm^{-(s-q/2)}\|\mathbf{a}\|_{\ell^2}.$$

Thus, $\mathcal{I}$ is a positive definite matrix. In view of the Raleigh-Ritz theorem [4, Theorem 4.2.2, p. 176], the lowest eigenvalue of this matrix is at least $cm^{-(s-q/2)}$. This implies (4.45). $\square$

Although not strictly a property of the kernels K_s, we find it convenient to record the following lemma, which will be needed in our proof of Theorem 2.2. This lemma is proved in much greater generality in [5, Theorem 3.2, Chapter 15].

Lemma 4.3 *Let $1 \le p \le \infty$, $s' > 0$. Then for any $c > 0$, the set $B_{c,s',p} := \{f \in L^p \ : \ \sup_{n\ge 1} 2^{ns'}E_{n,p}(f) \le c\}$ is compact in L^p.*

4.3 Background on approximation theory

The proof of Theorem 2.2 depends upon a number of facts from classical approximation theory, as well as some recent developments. In this section, we review the necessary facts.

First, for integer $r \geq 1$, the modulus of smoothness $\omega_r(f, \delta)$ of a 2π–periodic univariate function $f \in L^p([-\pi, \pi])$ is defined by first defining the forward difference operator

$$\Delta_t^r f(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k f(x + kt),$$

and setting

$$\omega_r(f, \delta) := \max_{|t| \leq \delta} \|\Delta_t^r f\|_p.$$

If $f \in L^p([-\pi, \pi]^q)$ and $\mathbf{r} = (r_1, \cdots, r_q) \geq 0$, $\mathbf{r} \neq (0, \cdots, 0)$, is a multi–integer, then the modulus of smoothness is defined in [19, Section 3.4.34] by

$$\omega_{\mathbf{r}}(f, \mathbf{h}) = \max_{|t_1| \leq h_1, \cdots, |t_q| \leq h_q} \left\| \left(\prod_{k=1}^{q} \Delta_{t_k,k}^{r_k} \right) f \right\|_p,$$

where the notation $\Delta_{t_k,k}^{r_k}$ means that the operator $\Delta_{t_k}^{r_k}$ is applied to the k-th variable in the argument of f and $\Delta_{t_k,k}^{0} f$ means that no difference is taken with respect to th k–th variable. We will write $\mathbf{e}_k$ to denote the vector in $\mathbb{R}^q$ with k-th coordinate equal to 1 and the remaining coordinates equal to 0.

For an integer $n \geq 0$, the class of (rectangular) trigonometric polynomials of order at most n is defined by

$$\mathbb{H}_n^{q,R} := \left\{ \sum_{\mathbf{k} \in \mathbb{Z}^q,\ |k_\ell| \leq n,\ \ell = 1, \cdots, q} a_{\mathbf{k}} \exp(i\mathbf{k} \cdot (\circ)) \ : \ a_{\mathbf{k}} \in \mathbb{C} \right\}.$$

For $f \in L^p$, the degree of approximation from $\mathbb{H}_n^{q,R}$ is defined by

$$E_{n,p;R}(f) = \inf_{P \in \mathbb{H}_n^{q,R}} \|f - P\|_p.$$

Let $r \geq 0$ be an integer, $\alpha \in (0, 1]$.It is proved in [19, Section 3.6.4] that for $f \in L^p$, the relation

$$\sup_{n \geq 1} n^{r+\alpha} E_{n,p;R}(f) < \infty \tag{4.47}$$

holds if and only if f has almost everywhere defined partial derivatives $D_k^r f$ satisfying

$$\max_{1 \leq k \leq q} \sup_{\delta > 0} \delta^{-\alpha} \omega_{2\mathbf{e}_k}(D_k^r f, \delta \mathbf{e}_k) < \infty. \tag{4.48}$$

(The formulation in [19] is not quite precise. However, the version which we have stated can be obtained using the same ideas as in [19]. See [20, 4.5.6] for an analogous statement in a slightly different context.) Since $\mathbb{H}_n^q \subseteq \mathbb{H}_n^{q,R} \subseteq \mathbb{H}_{\sqrt{q}n}^q$, we have

$$E_{\sqrt{q}n,p}(f) \leq E_{n,p;R}(f) \leq E_{n,p}(f).$$

Thus, (4.47) is equivalent to

$$\sup_{n \geq 1} n^{r+\alpha} E_{n,p}(f) < \infty.$$

We summarize these observations in the following proposition.

Proposition 4.4 *Let $1 \leq p \leq \infty$, and $f \in L^p$, $r \geq 0$ be integer, and $\alpha \in (0, 1]$. Then*

$$\sup_{n \geq 1} n^{r+\alpha} E_{n,p}(f) < \infty \tag{4.49}$$

if and only if f has almost everywhere defined partial derivatives $D_k^r f$ satisfying (4.48).

Next, we recall some results from the theory of algebraic polynomial approximation. Let Π_r^q denote the set of all algebraic polynomials of coordinatewise degree at most r. We wish to construct an approximation to a continuous function f on $[-1,1]^q$, defined analogously to (2.1), based on an arbitrary data set $\mathcal{C}\subset[-1,1]^q$. The mesh norm $\delta_{\mathcal{C}}:=\delta(\mathcal{C},[-1,1]^q)$ is defined analogously to (2.8). We divide $[-1,1]^q$ into equal subcubes of side $2\delta_{\mathcal{C}}$; the set of these subcubes will be denoted, in this part of the discussion only, by $\mathcal{R}_{\mathcal{C}}$. Each of the subcubes has at least one point of $\mathcal{C}$. We form a subset $\mathcal{C}_1\subset\mathcal{C}$ by choosing exactly one point ξ in each $R_\xi\in\mathcal{R}_{\mathcal{C}}$. Then it is clear that $\delta_{\mathcal{C}_1}\sim\delta_{\mathcal{C}}$. In the following discussion, the points in $\mathcal{C}\setminus\mathcal{C}_1$ do not play any role, and accordingly, we rename $\mathcal{C}_1$ to be $\mathcal{C}$. The following proposition follows from [9, Theorem 3.1], by taking the functional $P\to\int_{[-1,1]^q}P(\mathbf{x})d\mathbf{x}$ in place of γ in that theorem.

Proposition 4.5 *Let $\mathcal{C}$, $\mathcal{R}_{\mathcal{C}}$ be as above, $r\ge 1$ be an integer. There exists a constant $\gamma:=\gamma(r,q)$ with the following property. If $\delta_{\mathcal{C}}\le\gamma$, then*

$$\sum_{\xi\in\mathcal{C}}\mathrm{vol}_q(R_\xi)|P(\xi)|\sim\int_{[-1,1]^q}|P(\mathbf{x})|d\mathbf{x},\qquad P\in\Pi_{2r}^q. \tag{4.50}$$

Further, there exist real numbers $\{a_\xi\ :\ \xi\in\mathcal{C}\}$, such that

$$|a_\xi|\le c\,\mathrm{vol}_q(R_\xi)\le c\delta_{\mathcal{C}}^q,\qquad \xi\in\mathcal{C}, \tag{4.51}$$

and

$$\sum_{\xi\in\mathcal{C}}a_\xi P(\xi)=\int_{[-1,1]^q}P(\mathbf{x})d\mathbf{x},\qquad P\in\Pi_{2r}^q. \tag{4.52}$$

In this section only, let p_k denote the orthonormalized Chebyshev polynomial of degree k, with positive leading coefficient. We define

$$v_r(x,y)=\sum_{k=0}^{2r}p_k(x)p_k(y),\qquad x,y\in[-1,1],\ r=1,2,\cdots.$$

and extend this definition by writing

$$v_r(\mathbf{x},\mathbf{y})=\prod_{\ell=1}^{q}v_r(x_\ell,y_\ell),\qquad \mathbf{x},\mathbf{y}\in[-1,1]^q,$$

If $f:[-1,1]^q\to\mathbb{R}$ is continuous, $\mathcal{C}$ and $\{a_\xi\}$ are as in Proposition 4.5, we define

$$V_r(f,\mathbf{x})=\sum_{\xi\in\mathcal{C}}a_\xi f(\xi)v_r(\mathbf{x},\xi). \tag{4.53}$$

Using (4.51), (4.50), we conclude that

$$\|V_r(f)\|_{\infty,[-1,1]^q}\le c(r)\|f\|_{\infty,[-1,1]^q}. \tag{4.54}$$

In view of (4.52), $V_r(P)=P$ for all $P\in\Pi_r^q$. So, choosing $P^*\in\Pi_r^q$ with $\|f-P^*\|_{\infty,[-1,1]^q}\le 2\inf_{P\in\Pi_r^q}\|f-P\|_{\infty,[-1,1]^q}$, (4.54) yields

$$\|f-V_r(f)\|_{\infty,[-1,1]^q}=\|f-P^*-V_r(f-P^*)\|_{\infty,[-1,1]^q}\le c\|f-P^*\|_{\infty,[-1,1]^q}\le c\inf_{P\in\Pi_r^q}\|f-P\|_{\infty,[-1,1]^q}. \tag{4.55}$$

Using the direct theorem of approximation theory [19, Section 5.3.1], we conclude that if f has continuous partial derivatives of order up to r, then

$$\|f-V_r(f)\|_{\infty,[-1,1]^q}\le c(r)\sum_{k=1}^{q}\omega_{2\mathbf{e}_k}(D_k^rf,\mathbf{e}_k/r), \tag{4.56}$$

where the modulus of smoothness is defined analogously to (4.48), except that the maximum is taken for only those values of $t_1, \cdots, t_k$ which don't take the argument out of the cube in question. We note again that the operator V_r is determined entirely by the values $\{f(\xi)\}_{\xi\in\mathcal{C}}$.

We end this section by recording another observation, establishing a connection between the discrete norm used in the statement of the minimization problem (2.11) and the continuous L^p norm, which will be needed in the proof of Theorem 2.1(b).

Lemma 4.4 *For integer $n \ge 1$, $1 \le p \le \infty$, and $T \in \mathbb{H}_n^q$, we have*

$$\left\{\frac{1}{n^q}\sum_{0\le\mathbf{k}\le 3n-1}|T(2\pi\mathbf{k}/(3n))|^p\right\}^{1/p} \sim \|T\|_p. \tag{4.57}$$

Proof. When $q = 1$, (4.57) is the classical Marcinkiewicz–Zygmund inequality [21, Chapter X, Theorems 7.5, 7.28]. If $T \in \mathbb{H}_n^q$, then $T \in \mathbb{H}_{(n,\cdots,n)}$. So, in the case when $q > 1$, one obtains (4.57) by applying its univariate version to each of the variables separately. $\square$

5 Proofs of the main results in Section 2.

Our proof of Theorem 2.1(a) relies upon the next lemma, proved in [8, Theorem 2.1].

Lemma 5.1 *Let X be a normed linear space, $V \subset X$ be a finite dimensional subspace of X, X^* be the dual space of X, $\{x_j^*\}_{j=1}^M \subset X^*$, and Z_* be the span of $\{x_j^*\}_{j=1}^M$. Suppose that the restriction map $S : z^* \in Z_* \mapsto z^*|_V$ is injective, and the operator norm $\|S^{-1}\| \le \kappa$ for some $\kappa > 0$. Then for every $f \in X$ and $\kappa_1 > \kappa$, there exists $\mathsf{T}(f) \in V$ such that*

$$z^*(\mathsf{T}(f)) = z^*(f) \qquad \text{for every } z^* \in Z_*, \tag{5.1}$$

and

$$\|f - \mathsf{T}(f)\|_X \le (1+\kappa_1)\inf_{v\in V}\|f - v\|_X. \tag{5.2}$$

We will use this lemma with W_s^p in place of X, $\mathbb{H}_{N^*}^q$ in place of V for a suitable N^*, and point evaluation functionals in place of x_j^*'s.

Proof of Theorem 2.1. The proof of this theorem is similar to that of [8, Theorem 3.1], except that the details are much more complicated, requiring the use of Theorem 4.2 and Theorem 4.4. In this proof only, let $X = W_s^p$, $x_j^*(f) = f(\mathbf{y}_j)$, $j = 1, \cdots, M$. Since $s > q/p$, Proposition 2.1 implies that $x_j^* \in X^*$, $j = 1, \cdots, M$. Let $\{a_j\}_{j=1}^M \subset \mathbb{R}$ and $z^* = \sum_{j=1}^M a_j x_j^*$. Let N^* be as in Theorem 4.4, and $V = \mathbb{H}_{N^*}^q$. To estimate $\|S^{-1}\|$ for the operator S as in Lemma 5.1, we need to find $T \in \mathbb{H}_{N^*}^q$ for a suitable N^*, and estimate $|z^*(T)|/\|T\|_{W_s^p}$ from below. Let f^* be chosen so that $\|z^*\|_{X^*} \le (4/3)|z^*(f^*)|$ and $\|f^*\|_{W_s^p} = 1$. We will prove that $\sigma_{N^*}(h, f^*) \in \mathbb{H}_{N^*}^q$ (cf. (4.16)) satisfies

$$\sup_{T\in V}\frac{|z^*(T)|}{\|T\|_{W_s^p}} \ge \frac{|z^*(\sigma_{N^*}(h,f^*))|}{\|\sigma_{N^*}(h,f^*)\|_{W_s^p}} \ge c\|z^*\|_{X^*}. \tag{5.3}$$

The part (a) of the theorem will then follow from Lemma 5.1.

We start by observing that for $f \in W_s^p$, (2.5) shows that

$$z^*(f) = \frac{1}{(2\pi)^q}\int_{[-\pi,\pi]^q}\left\{\sum_{j=1}^M a_j K_s(\mathbf{y}-\mathbf{y}_j)\right\}f^{(s)}(\mathbf{y})d\mathbf{y} = \frac{1}{(2\pi)^q}\int_{[-\pi,\pi]^q}G(\mathbf{y})f^{(s)}(\mathbf{y})d\mathbf{y},$$

where G is defined as in Theorem 4.4. In light of the duality principle and the definition (2.3), we see that

$$\|z^*\|_{X^*} = \sup\{|z^*(f)| \ : \ \|f\|_{W_s^p} = 1\} = \|G\|_{p'}. \tag{5.4}$$

Further a comparison of Fourier coefficients implies that for any integer n,

$$\begin{aligned} z^*(\sigma_n(h,f)) &= \frac{1}{(2\pi)^q}\int_{[-\pi,\pi]^q} G(\mathbf{y})(\sigma_n(h,f))^{(s)}(\mathbf{y})d\mathbf{y} = \frac{1}{(2\pi)^q}\int_{[-\pi,\pi]^q} G(\mathbf{y})\sigma_n(h,f^{(s)},\mathbf{y})d\mathbf{y} \\ &= \frac{1}{(2\pi)^q}\int_{[-\pi,\pi]^q} \sigma_n(h,G,\mathbf{y})f^{(s)}(\mathbf{y})d\mathbf{y} \end{aligned}$$

Then

$$\begin{aligned} |z^*(f^*) - z^*(\sigma_{N^*}(h,f^*))| &= \left|\frac{1}{(2\pi)^q}\int_{[-\pi,\pi]^q} (G(\mathbf{y}) - \sigma_{N^*}(h,G,\mathbf{y}))\, {f^*}^{(s)}(\mathbf{y})d\mathbf{y}\right| \\ &\le \|G - \sigma_{N^*}(h,G)\|_{p'} \le (1/2)\|G\|_{p'} = (1/2)\|z^*\|_{X^*} \le (2/3)|z^*(f^*)|. \end{aligned}$$

Therefore,

$$|z^*(\sigma_{N^*}(h,f^*))| \ge (1/3)|z^*(f^*)| \ge (1/4)\|z^*\|_{X^*}. \tag{5.5}$$

Moreover, (4.17) implies that

$$\|\sigma_{N^*}(h,f^*)\|_{W^p_s} = \|(\sigma_{N^*}(h,f^*))^{(s)}\|_p = \|\sigma_{N^*}(h,{f^*}^{(s)})\|_p \le c\|{f^*}^{(s)}\|_p = c. \tag{5.6}$$

The estimate (5.3) follows from (5.5) and (5.6).

We note that necessarily, $\|\mathbf{P}(f)\|_{W^p_s} \le \|f\|_{W^p_s}$. Therefore, part (b) is a simple consequence of Lemma 4.4. □

Proof of Theorem 2.2.

To prove part (a), we observe that in view of (2.6) and the fact that $\|\mathbb{P}^*_n\|_{W^p_s} \le c\|f\|_{W^p_s}$ for all n, the sequence $\{\mathbb{P}^*_n\} \subset B_{c,s-q/p,\infty}$ for a suitable constant c. Let Λ_1 be any subsequence of Λ. Then Lemma 4.3 shows that the sequence $\{\mathbb{P}^*_n\}_{n\in\Lambda_1}$ has a subsequence $\{\mathbb{P}^*_n\}_{n\in\Lambda_2}$, which converges uniformly. Let P be the limit of this subsequence. We will show that if (2.12) is satisfied, then $P(\mathbf{x}_0) = f(\mathbf{x}_0)$. Let $\epsilon > 0$ be arbitrary. Since P and f are continuous on $[-\pi,\pi]^q$, there is $\tilde{\delta} > 0$ such that

$$|f(\mathbf{x}) - f(\mathbf{y})| \le \epsilon/3,\ |P(\mathbf{x}) - P(\mathbf{y})| \le \epsilon/3, \qquad \text{for all } \mathbf{x},\mathbf{y} \in [-\pi,\pi]^q,\ \|\mathbf{x}-\mathbf{y}\| \le \tilde{\delta}.$$

Further, there exists N so that $n \ge N$, $n \in \Lambda_2$ imply that $\|P - \mathbb{P}^*_n\|_\infty \le \epsilon/3$. In view of (2.12), there exists $n \in \Lambda_2$, $n \ge N$ such that some point $\mathbf{y}_{j,n} \in Y$ satisfies $\|\mathbf{y}_{j,n} - \mathbf{x}_0\| \le \tilde{\delta}$. Then $f(\mathbf{y}_{j,n}) = \mathbb{P}^*_n(\mathbf{y}_{j,n})$, and we have

$$\begin{aligned} |f(\mathbf{x}_0) - P(\mathbf{x}_0)| &\le |f(\mathbf{x}) - f(\mathbf{y}_{j,n})| + |f(\mathbf{y}_{j,n}) - \mathbb{P}^*_n(\mathbf{y}_{j,n})| + |\mathbb{P}^*_n(\mathbf{y}_{j,n}) - P(\mathbf{y}_{j,n})| + |P(\mathbf{y}_{j,n}) - P(\mathbf{x}_0)| \\ &\le \epsilon/3 + 0 + \|\mathbb{P}^*_n - P\|_\infty + \epsilon/3 \le \epsilon. \end{aligned}$$

Since this is true for every subsequential limit of $\mathbb{P}^*_n$, $n \in \Lambda$, this proves part (a).

To prove part (b), let r be an integer and $\alpha \in (0,1]$ be chosen so that $s - q/p = r + \alpha$. Since $\mathbb{P}^*_n, f \in B_{c,s-q/p,\infty}$, Proposition 4.4 implies that they both have r derivatives satisfying (4.48). In this proof only, let $\mathbb{P}(\mathbf{y}) = \mathbb{P}^*_n(\mathbf{x}_0 + \delta\mathbf{y})$, $\tilde{f}(\mathbf{y}) = f(\mathbf{x}_0 + \delta\mathbf{y})$, $\mathbf{y} \in [-1,1]^q$. Then the assumptions of part (b) ensure that we can construct the operator V_r as in (4.53) based on $Y_n \cap K$ in place of $\mathcal{C}$. In this proof only, if $k \in \{1,\cdots,q\}$, $F = D^r_k f$, then $D^r_k\tilde{f}(\mathbf{y}) = \delta^r F(\mathbf{x}_0 + \delta\mathbf{y})$. Further,

$$\Delta^2_{1/r,k}(D^r_k\tilde{f})(\mathbf{y}) = (\Delta^2_{\delta/r,k}F)(\mathbf{x}_0 + \delta\mathbf{y}).$$

Hence,

$$\|\Delta^2_{1/r,k}(D^r_k\tilde{f})\|_{\infty,[-1,1]^q} = \|\Delta^2_{\delta/r,k}F(\mathbf{x}_0 + \circ)\|_{\infty,[-\delta,\delta]^q} \le \omega_2(D^r_k f, (\delta/r)\mathbf{e}_k) \le c\delta^\alpha.$$

A similar estimate holdes also for $\mathbb{P}$ in place of $\tilde{f}$. Using (4.56) and the fact that $r + \alpha = s - q/p$, we deduce that

$$\|\tilde{f} - V_r(\tilde{f})\|_{\infty,[-1,1]^q} \le c\delta^{s-q/p},\ \|\mathbb{P} - V_r(\mathbb{P})\|_{\infty,K} \le c\delta^{s-q/p}.$$

We now observe that $V_r(\tilde{f}) = V_r(\mathbb{P})$, and hence, the above inequalities imply that $\|\tilde{f} - \mathbb{P}\|_{\infty,[-1,1]^q} \le c\delta^{s-q/p}$. Scaling back to the original scale, we obtain (2.13). □

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