Fast Memory Efficient Construction Algorithm for Hierarchically Semi-separable Representations

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Overview

- Review of Hierarchically Semi-Separable (HSS) Representation
  - Notation
- Previous HSS Algorithm Complexities
- Memory Efficient Algorithm
  - Phase 1
  - Phase 2
- Memory Consumption
- “A Fast Memory Efficient Construction Algorithm for Hierarchically Semi-Separable Representations” submitted for publication

Use a Partition Tree to Block Partition a Matrix

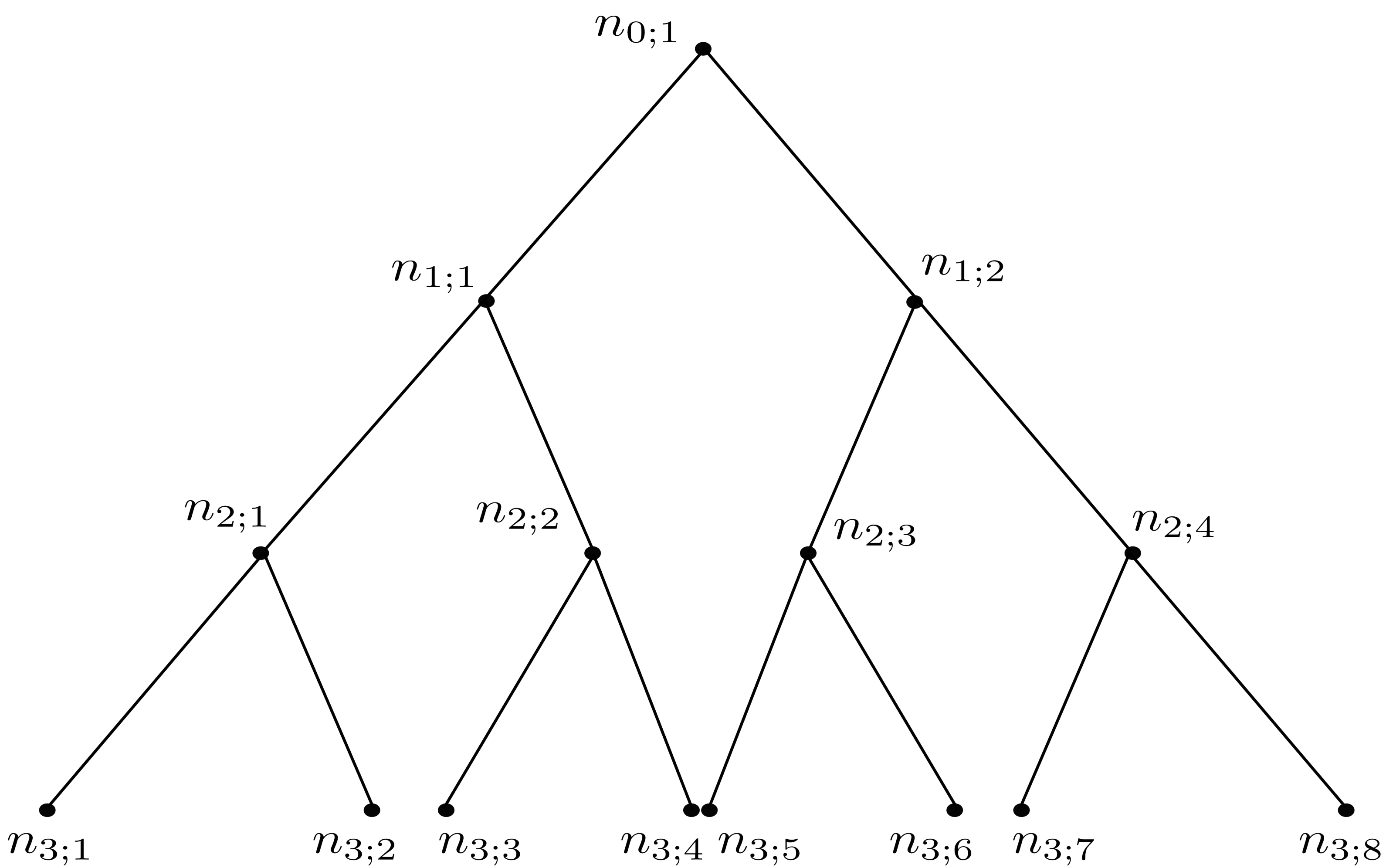
- Partition A according to the integers at the first level of the partition tree

$$\begin{array}{cc} & \begin{array}{cc} n_{1;1} & n_{1;2} \end{array} \\ A_{0;1,1} = \begin{array}{c} m_{1;1} \\ m_{1;2} \end{array} & \begin{pmatrix} A_{1;1,1} & A_{1;1,2} \\ A_{1;2,1} & A_{1;2,2} \end{pmatrix} \end{array}$$

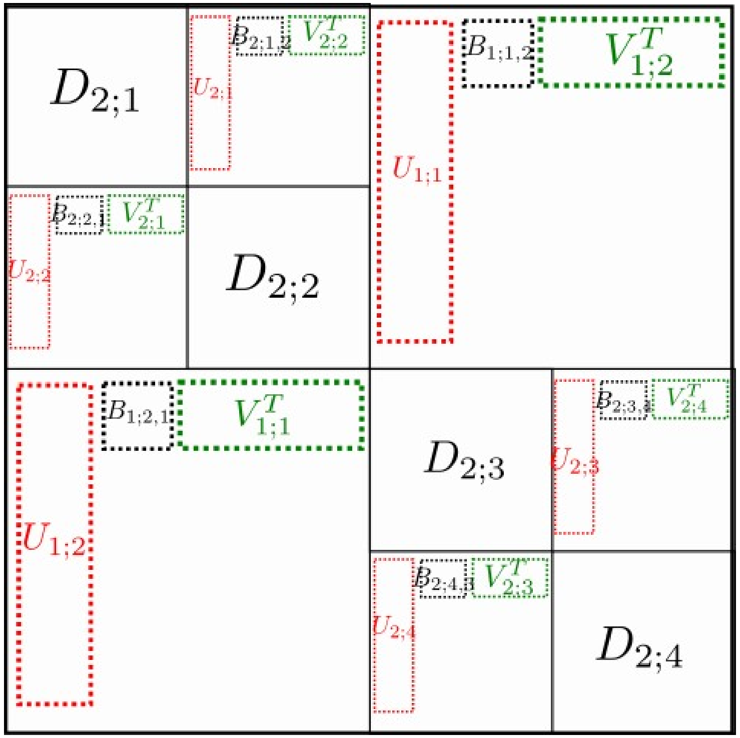
- Recursively partition the block rows and columns of A

$$\begin{array}{cc} & \begin{array}{cccc} n_{2;1} & n_{2;2} & n_{2;3} & n_{2;4} \end{array} \\ A_{0;1,1} = \begin{array}{c} m_{2;1} \\ m_{2;2} \\ m_{2;3} \\ m_{2;4} \end{array} & \begin{pmatrix} A_{2;1,1} & A_{2;1,2} & A_{2;1,3} & A_{2;1,4} \\ A_{2;2,1} & A_{2;2,2} & A_{2;2,3} & A_{2;2,4} \\ A_{2;3,1} & A_{2;3,2} & A_{2;3,3} & A_{2;3,4} \\ A_{2;4,1} & A_{2;4,2} & A_{2;4,3} & A_{2;4,4} \end{pmatrix} \end{array}$$

Definition – Complete Partition Tree

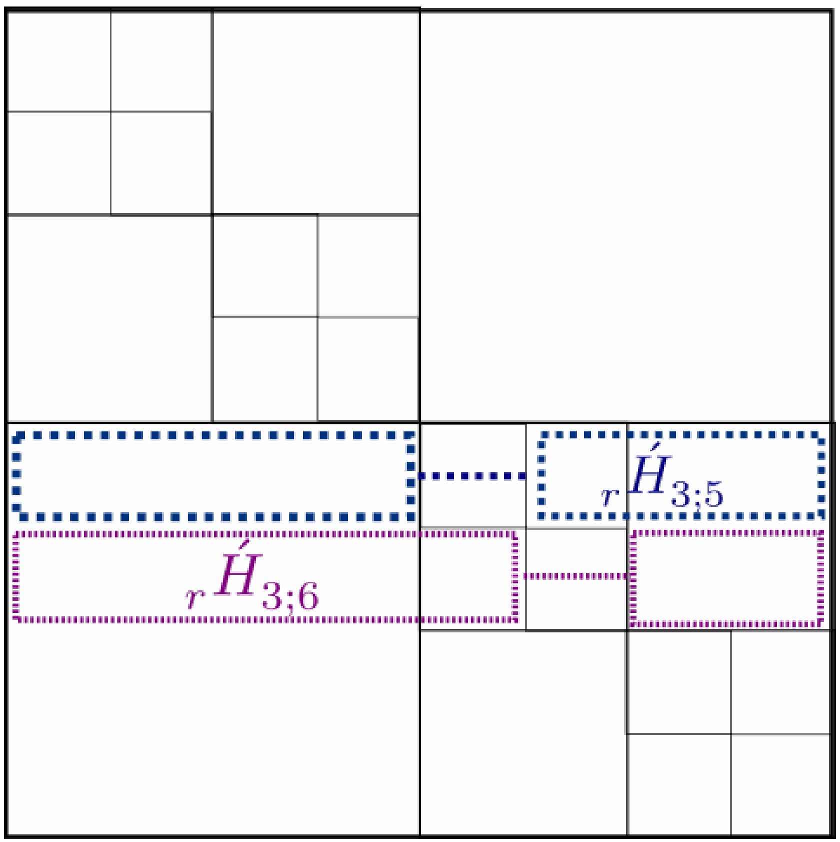


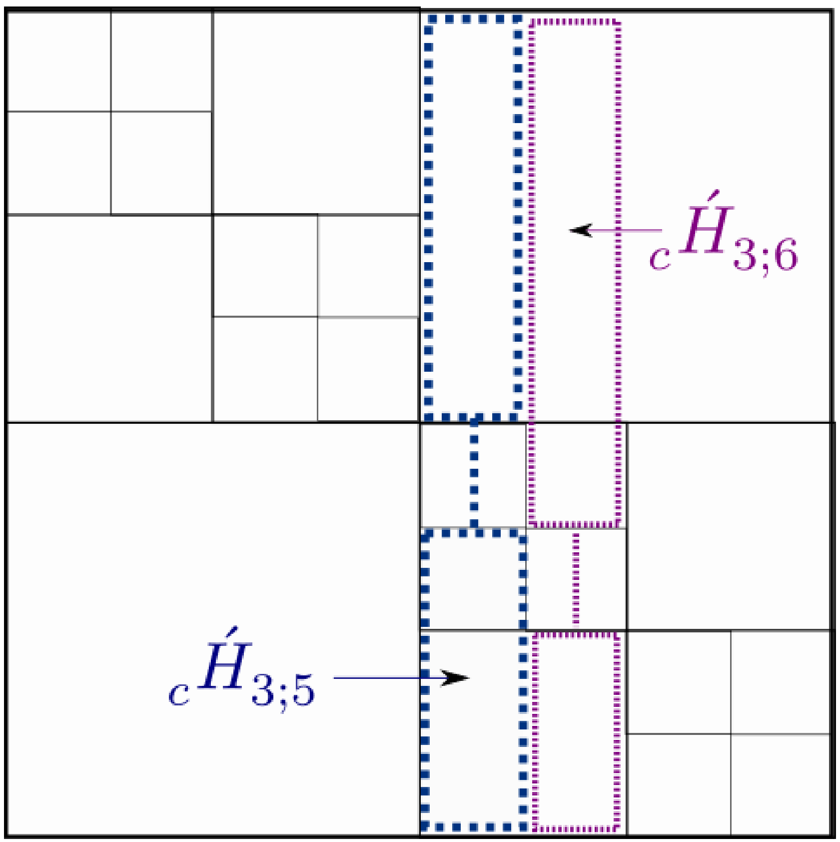
HSS Representation



- Off-Diagonal blocks thus have low rank and can be compressed
- Only store smaller basis matrices ($U_{k;i}, V_{k;i}$) and translation operators ($R_{k;i}, W_{k;i}$)

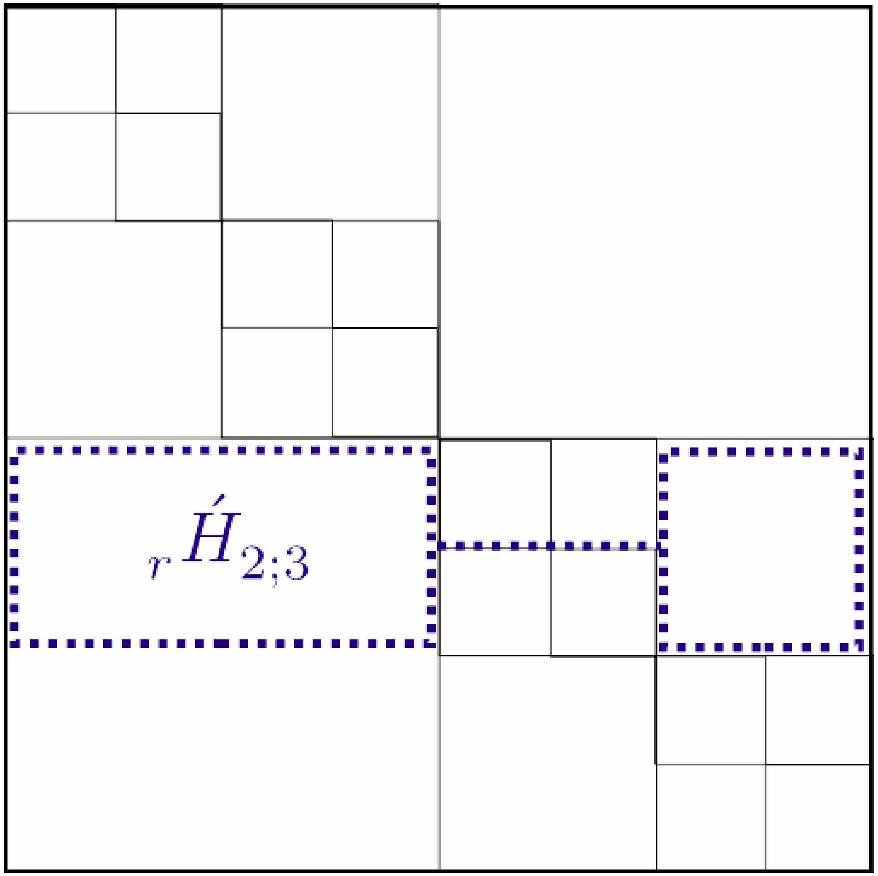
Definition - Hankel Blocks

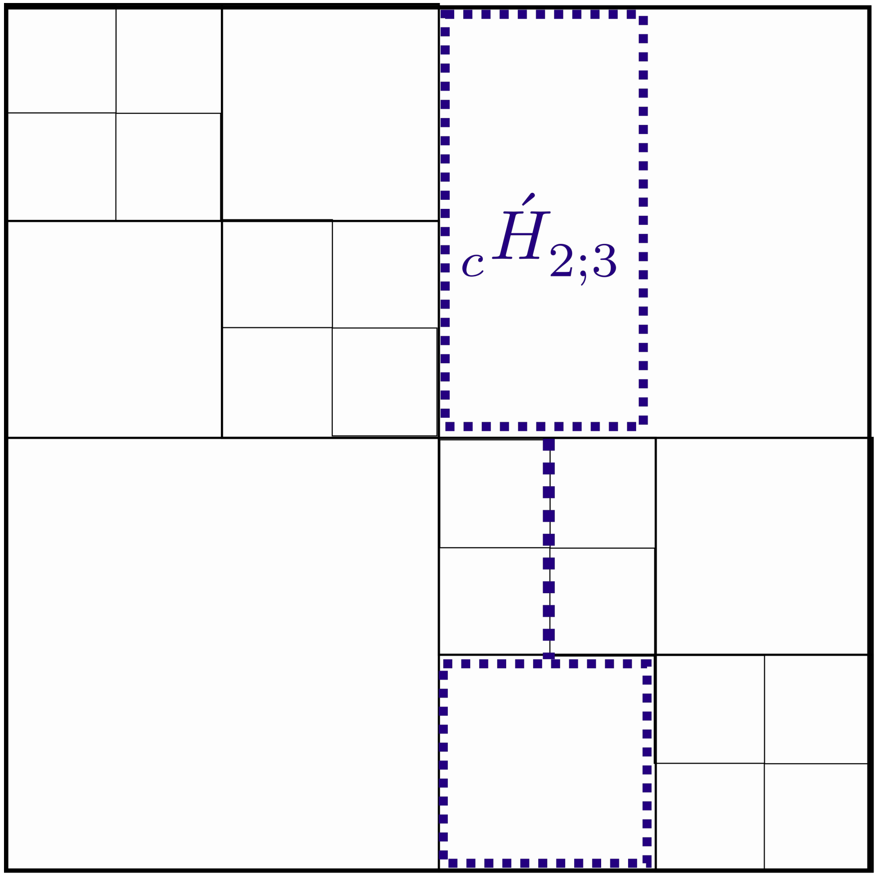




Block rows/columns of A, excluding diagonal blocks

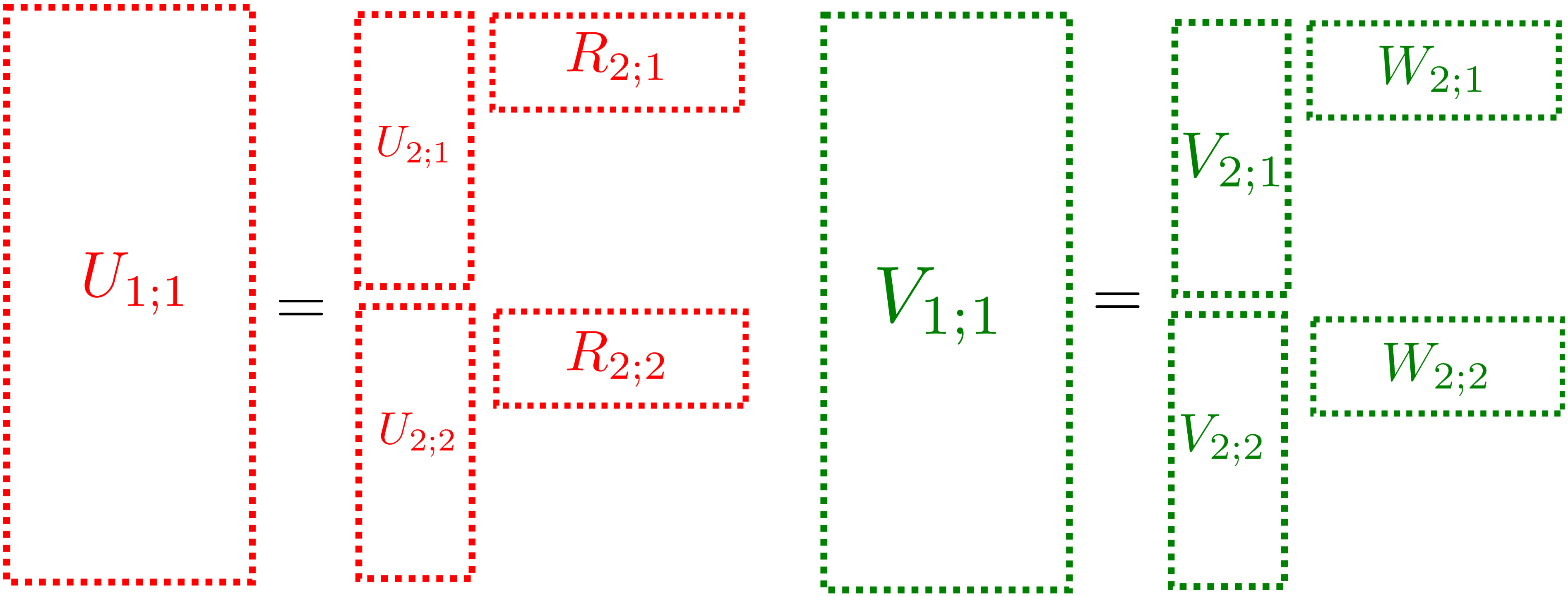
Definition - Hankel Blocks





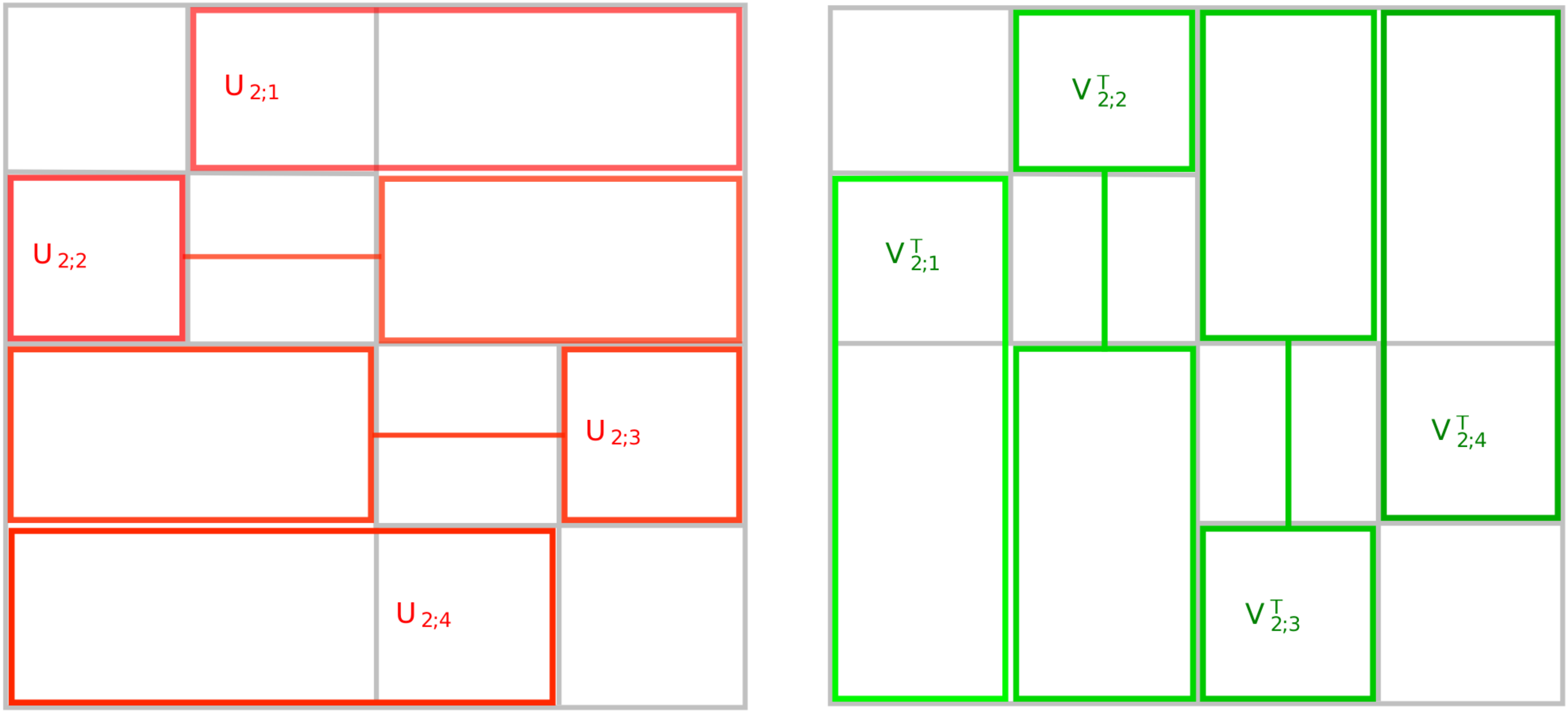
Block rows/columns of A, excluding diagonal blocks

Larger Basis Matrices Can Be Stored as Translated Versions of Smaller Basis Matrices

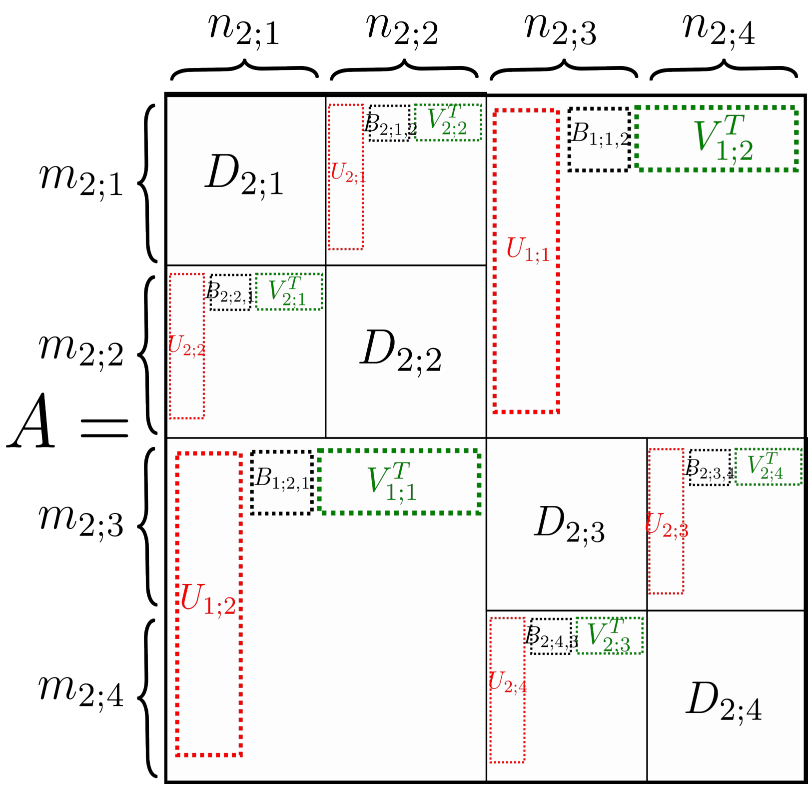


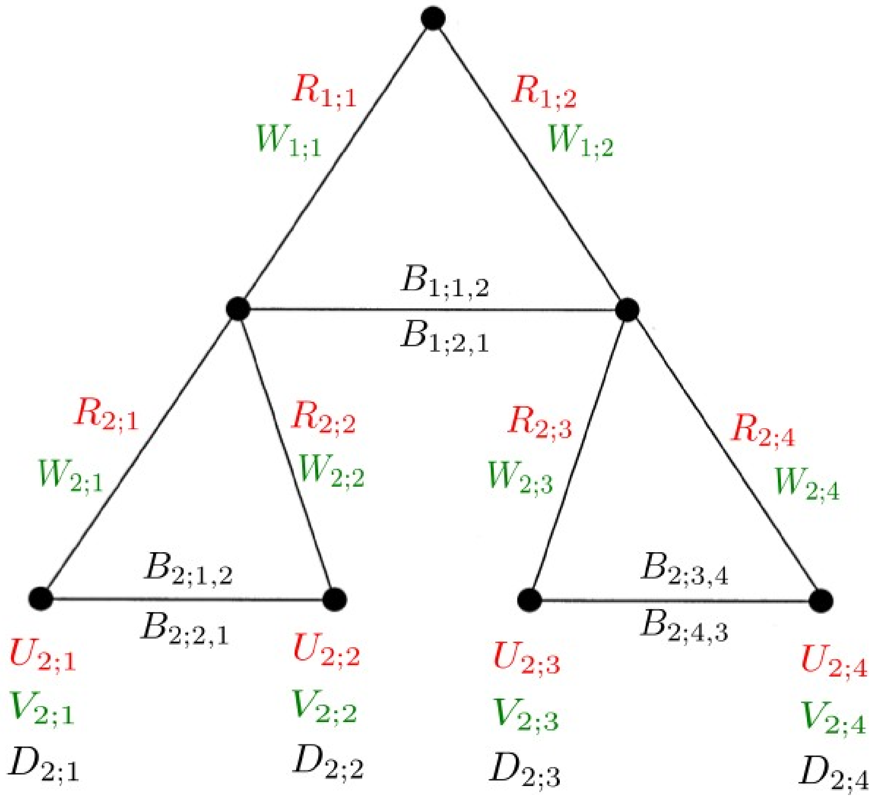
$$U_{1;2} = \begin{matrix} U_{2;3}\, R_{2;3} \\ U_{2;4}\, R_{2;4} \end{matrix} \qquad V_{1;2} = \begin{matrix} V_{2;3}\, W_{2;3} \\ V_{2;4}\, W_{2;4} \end{matrix}$$

Example 2 Level Column and Row Bases



Example 2 Level HSS Representation and Corresponding HSS Tree





Note: Notice the larger U's and V's are not stored, and do not appear in the HSS Tree

Definition - HSS Tree

- An HSS tree of a matrix is the corresponding partition tree decorated with $U_{k;i}, V_{k;i}, D_{k;i}, R_{k;i}, W_{k;i}$ and $B_{k;i,j}$.
  - The matrices $U_{k;i}, V_{k;i}, D_{k;i},$ are stored at each leaf node $(k;i)$.
  - The matrices $R_{k;i}$ and $W_{k;i}$ are stored at each edge which connects parent to child node, $(k;i)$.
  - We add edges to the partition tree from node $(k;i)$ to node $(k;j)$ corresponding to $B_{k;i,j}$.

Definition of HSS Representation

- If $(k;i)$ is a leaf node, $D_{k;i} = A_{k;i,i}$
- If $(k;i)$ is not a leaf node,

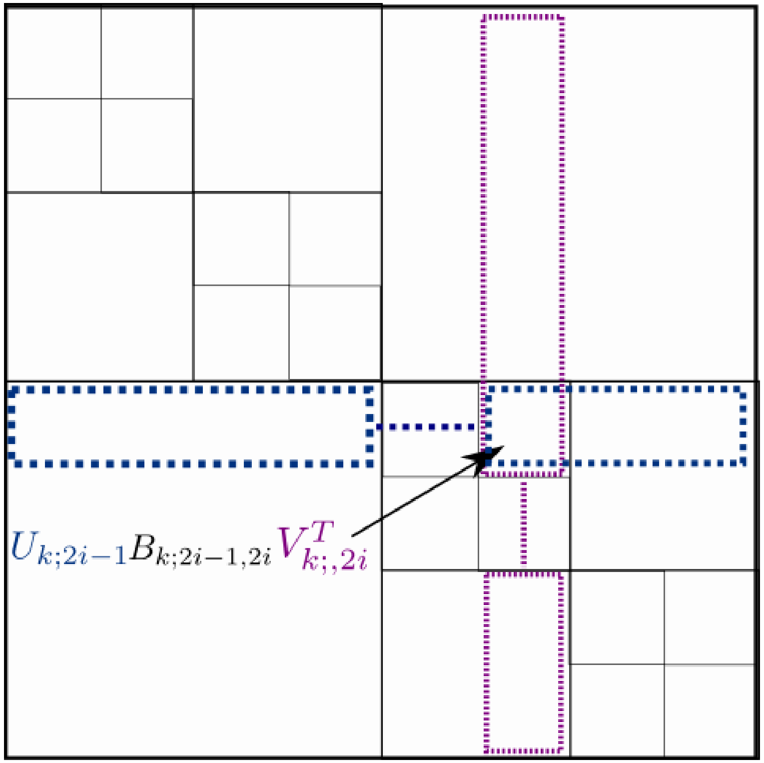
$$A_{k;2i-1,2i} = U_{k;2i-1} B_{k;2i-1;2i} V_{k;2i}^T,$$
$$A_{k;2i-1,2i} = U_{k;2i} B_{k;2i;2i-1} V_{k;2i-1}^T,$$

- Where,

$$U_{k;i} = \begin{pmatrix} U_{k+1;2i-1} R_{k+1;2i-1} \\ U_{k+1;2i} R_{k+1;2i} \end{pmatrix}, \quad V_{k;i} = \begin{pmatrix} V_{k+1;2i-1} W_{k+1;2i-1} \\ V_{k+1;2i} W_{k+1;2i} \end{pmatrix}.$$

An Inefficient Method to Compute the HSS Representation

- One obvious way to form the HSS representation of a matrix would be to take a Singular Value Decomposition (SVD) all Hankel blocks at each level of the HSS representation.
- This is extremely slow, $O(n^3)$ flops.
- Not memory efficient, $O(n^2)$ memory



A More Efficient Way to Compute the HSS Representation

- Previous HSS construction algorithms (Xia, Chandrasekeran, Martinsson) focused on speed, requiring $O(n^2)$ flops.
- It seems they were unaware they require $O(n^2)$ memory in the worst case.
- We present an HSS construction algorithm which requires $O(n^{1.5})$ peak workspace memory in the worst case, while still requiring only $O(n^2)$ flops.
- We require only $O(n \log n)$ memory for a complete tree.

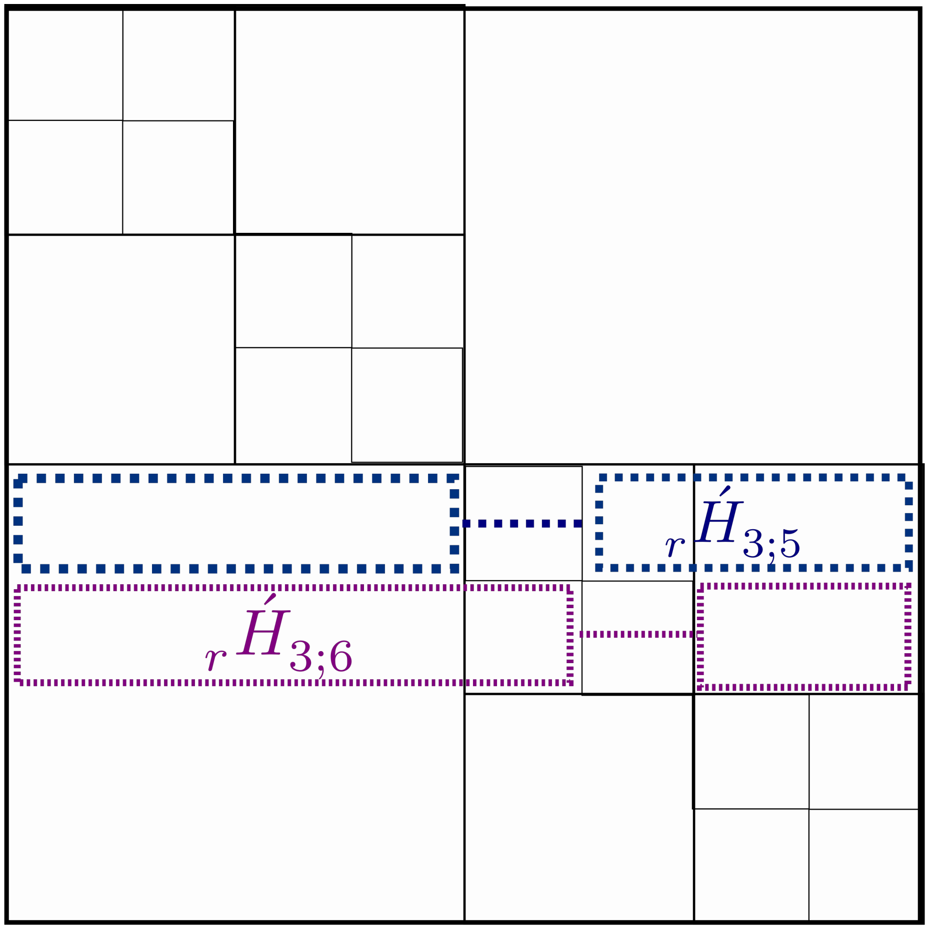
Main Points of this Talk

- Basic building blocks for $O(n^2)$ flop construction algorithm
- Peak memory consumption

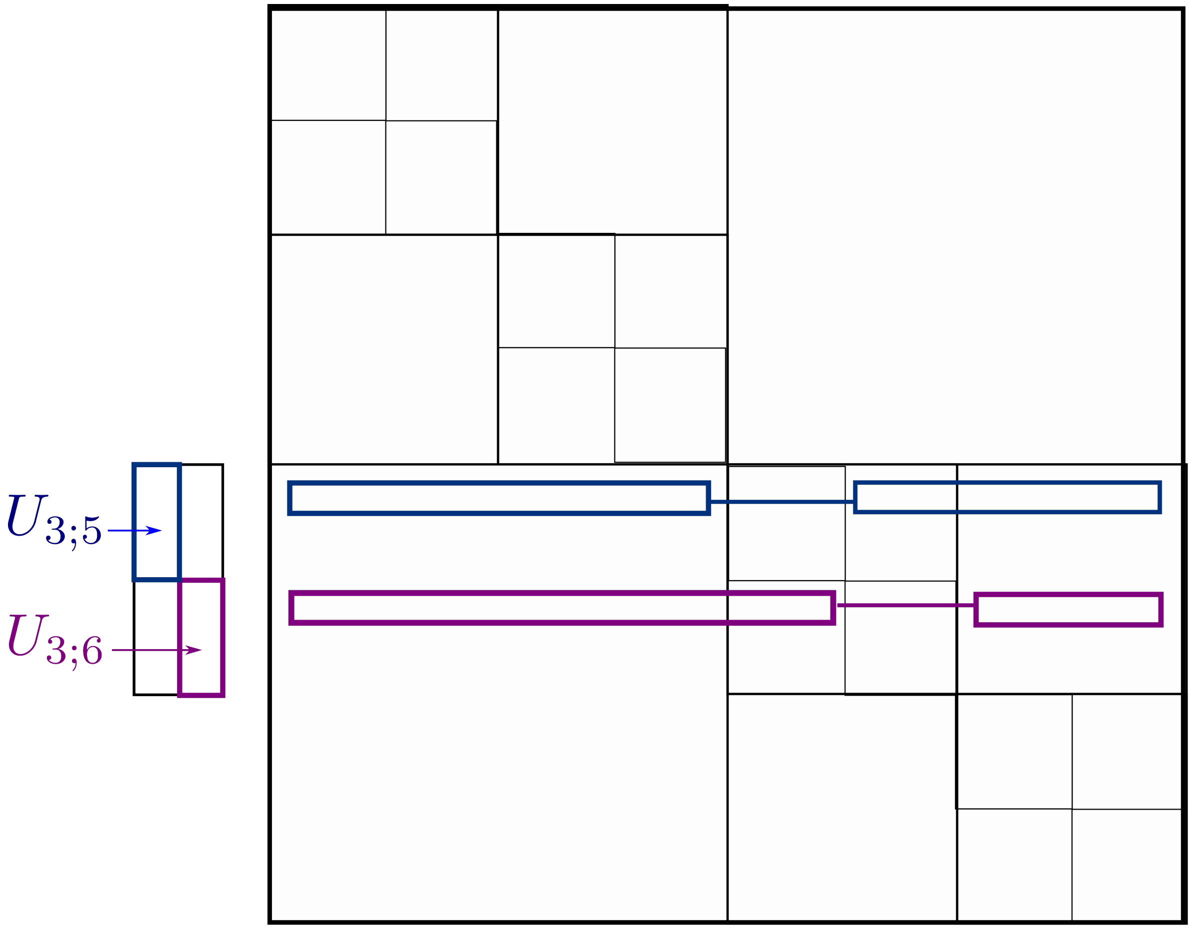
Phase 1 & Phase 2 of our Construction Algorithm

- Phase 1 – Computation of Basis Matrices, $U_{k;i}$, and $V_{k;i}$, as well as Translation Operators $R_{k;i}$ and $W_{k;i}$
- Phase 2 – Computation of Expansion Coefficients $B_{k;i,j}$

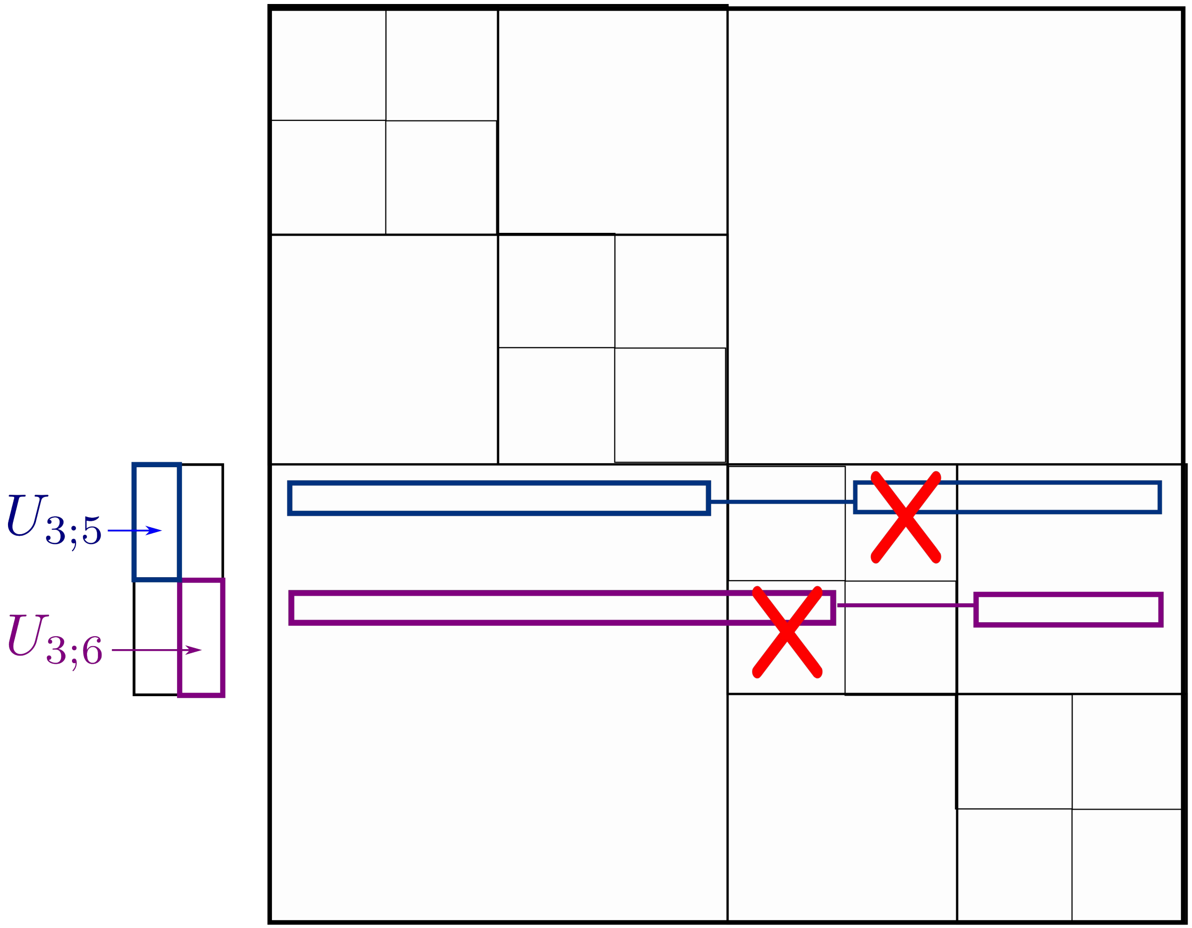
Phase 1 - Take SVD's Hankel Blocks at Leaf Nodes



Construction Algorithm - Phase 1



Construction Algorithm - Phase 1



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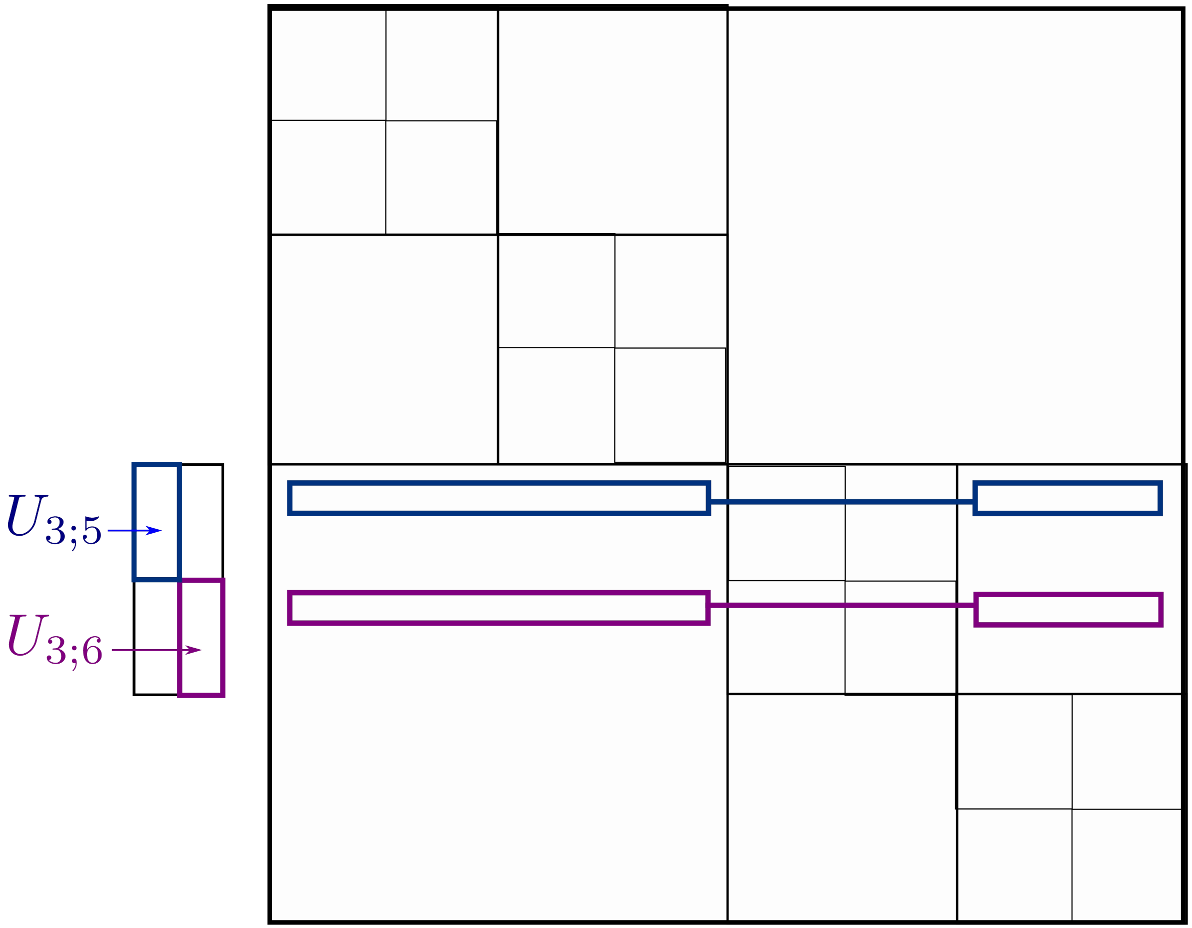
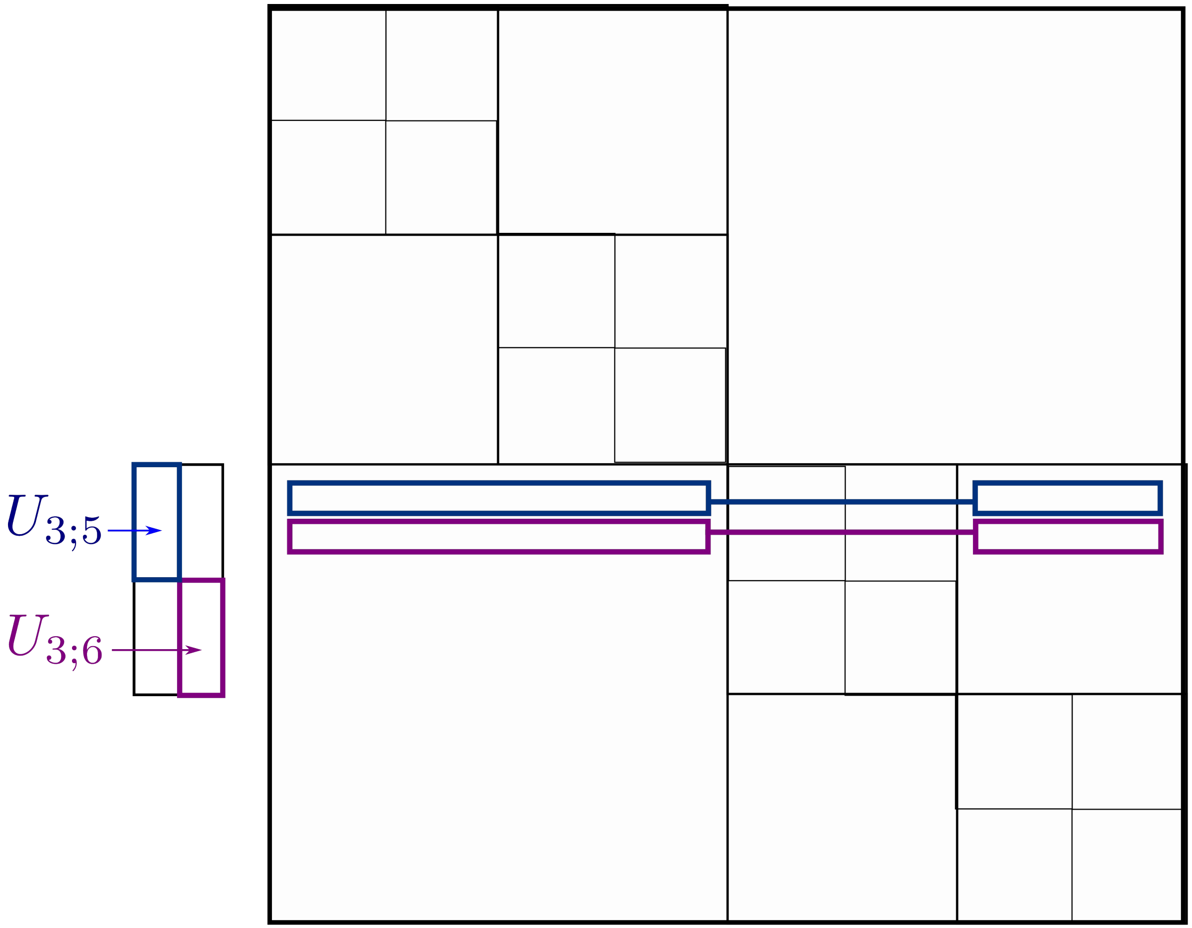
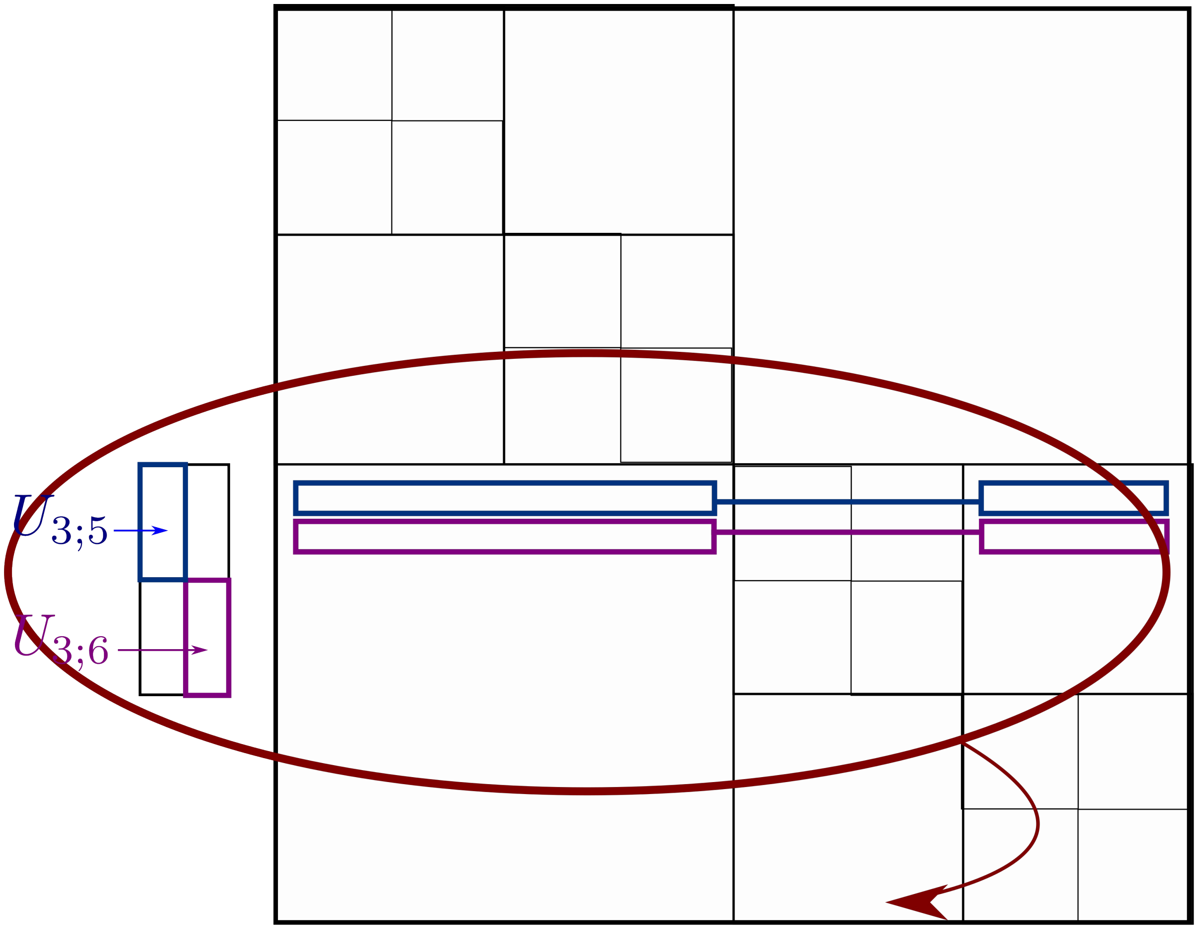


Construction Algorithm - Phase 1



Construction Algorithm - Phase 1



Construction Algorithm - Phase 1

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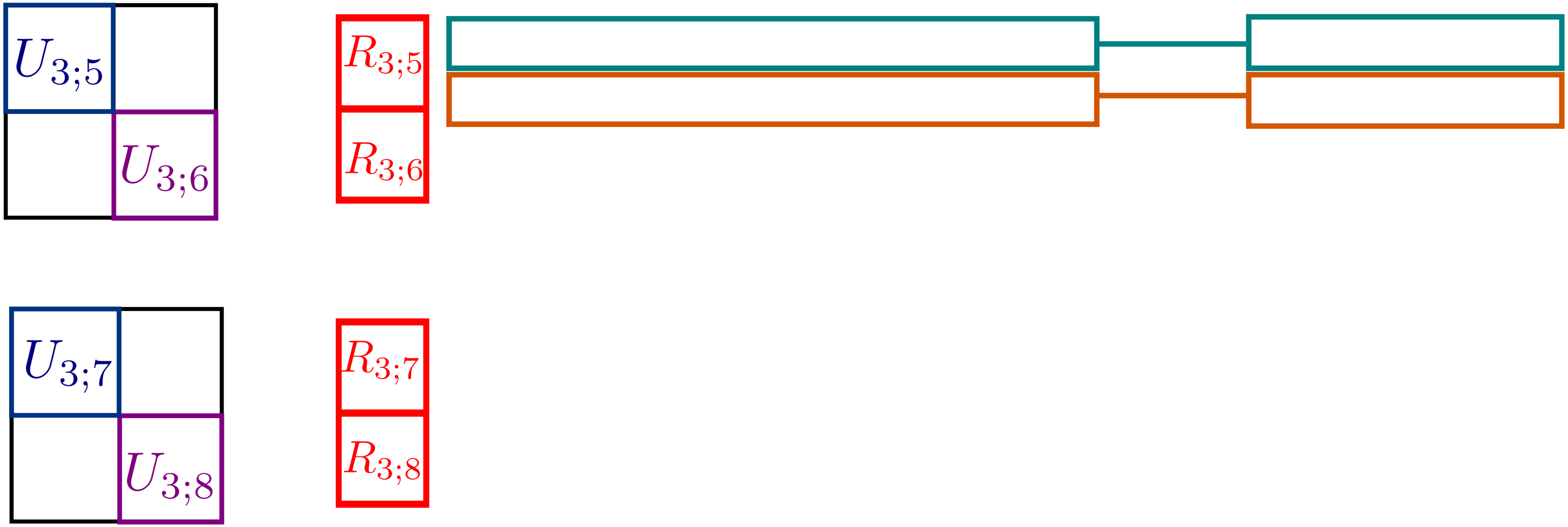
Construction Algorithm - Phase 1

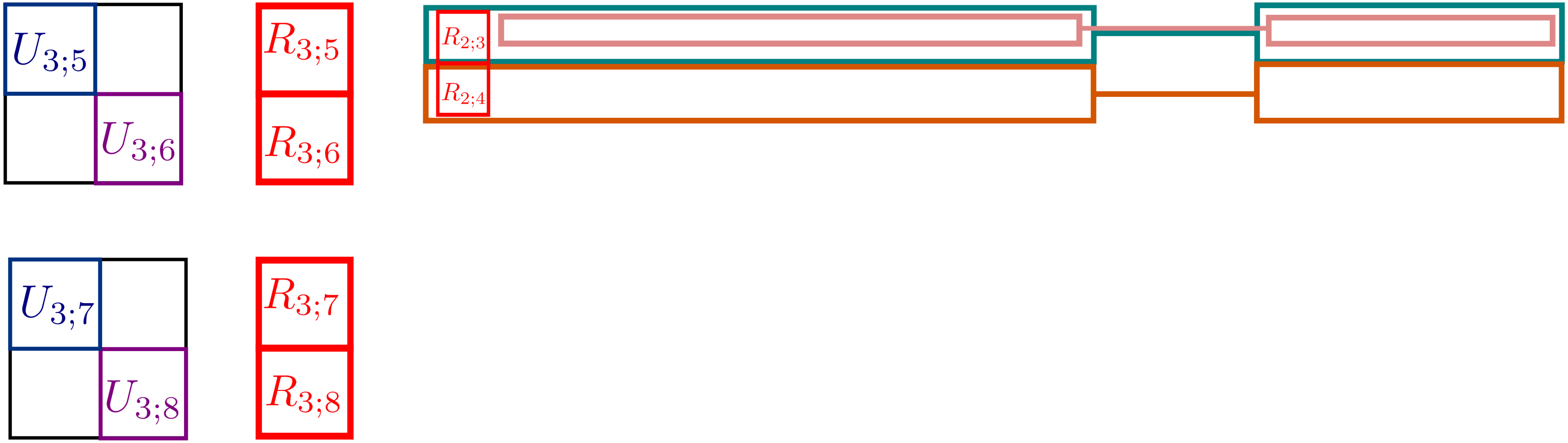

$U_{3;5}$
$U_{3;6}$
$R_{3;5}$
$R_{3;6}$


$U_{3;7}$
$U_{3;8}$
$R_{3;7}$
$R_{3;8}$

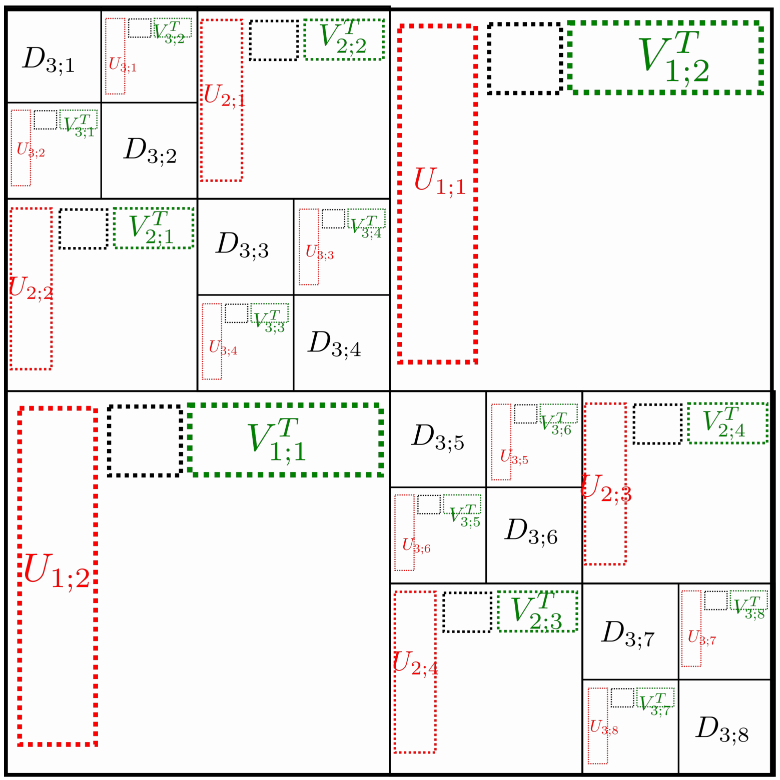
Construction Algorithm - Phase 1

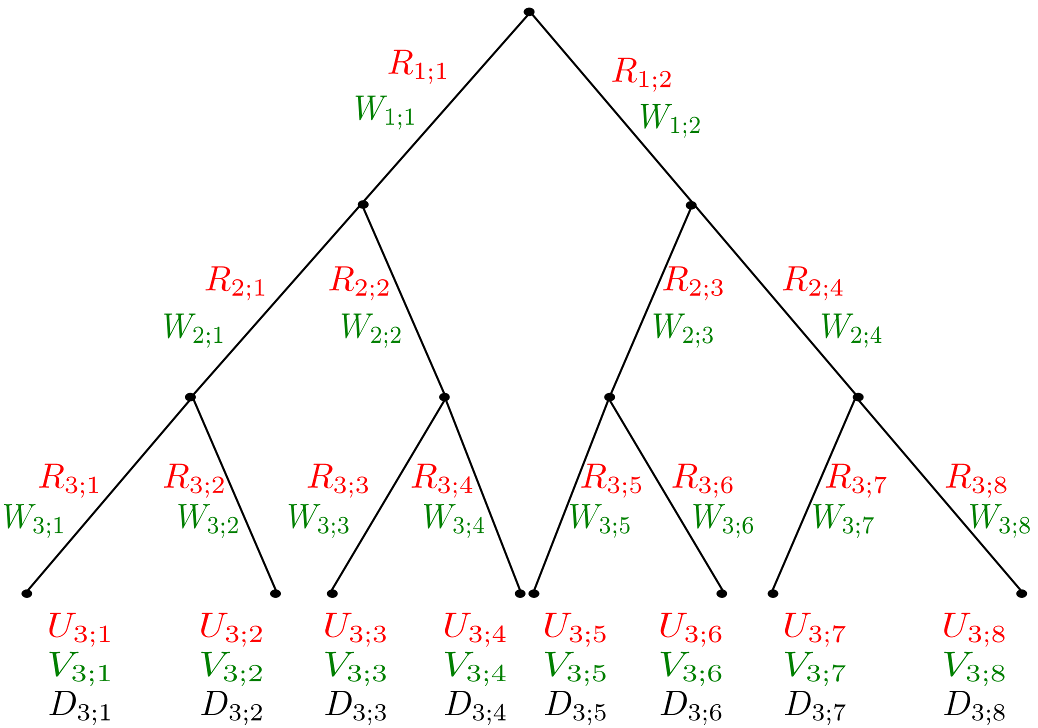


Construction Algorithm - Phase 1

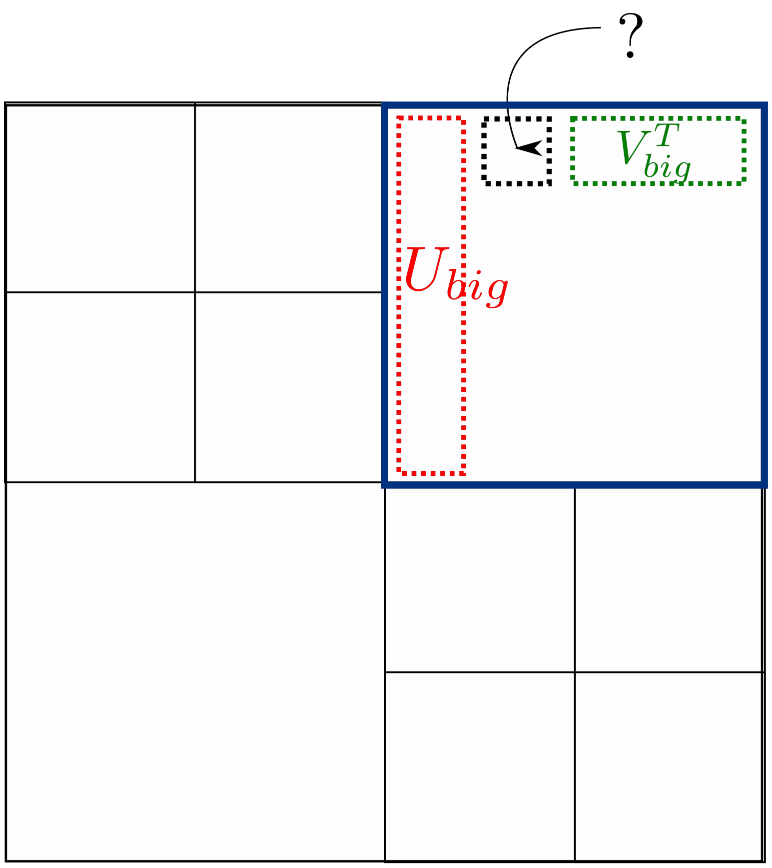


Phase 1 Complete





Construction Algorithm - Phase 2 – The Naive Way



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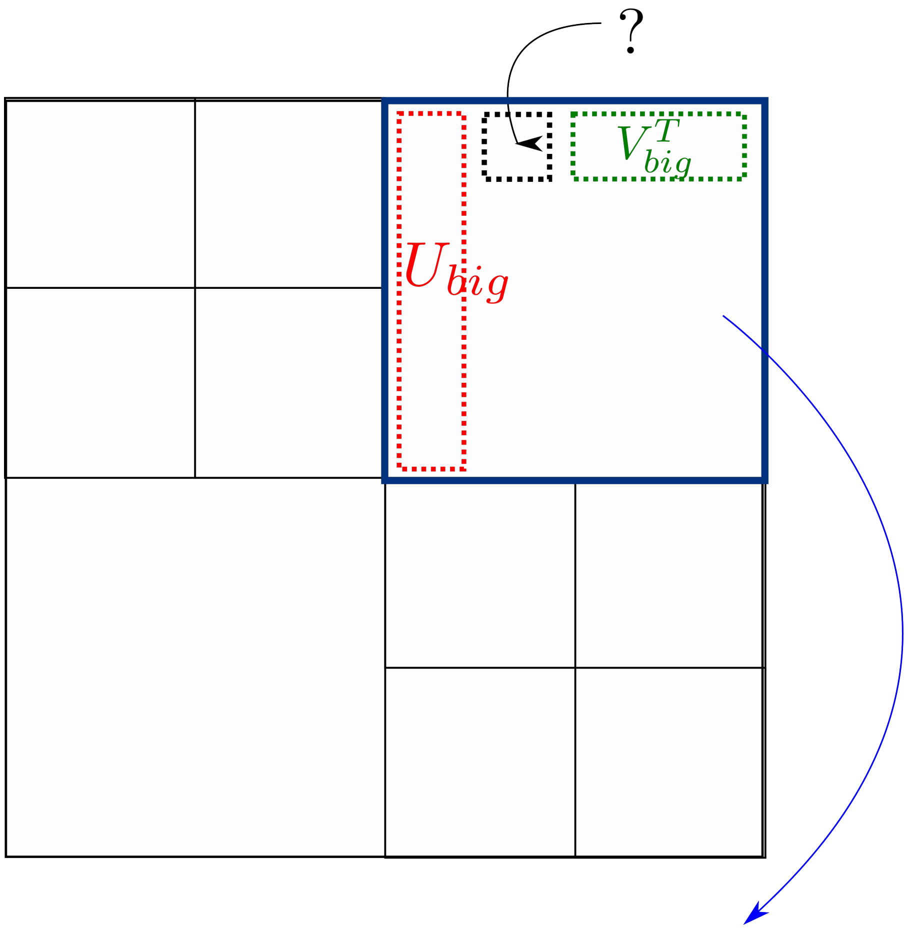

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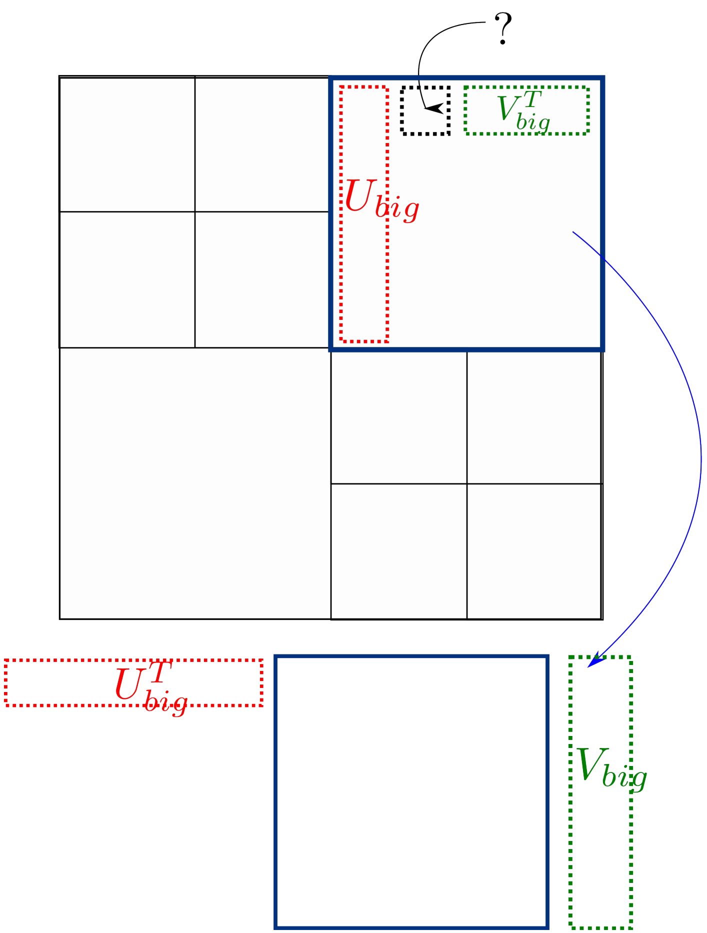

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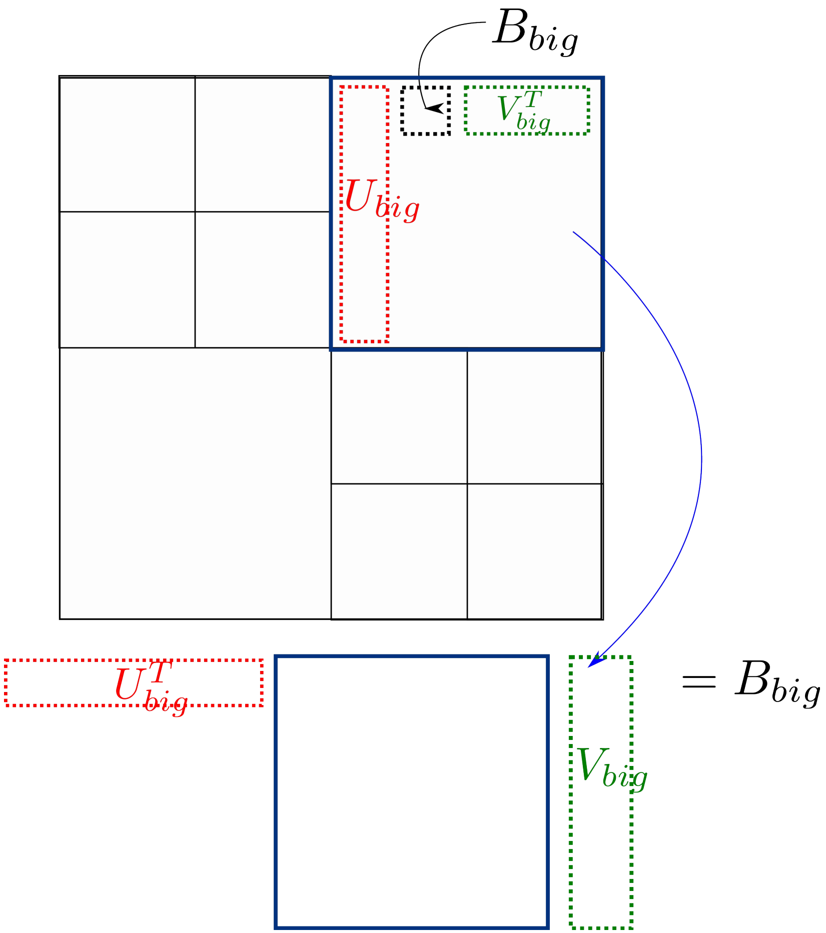

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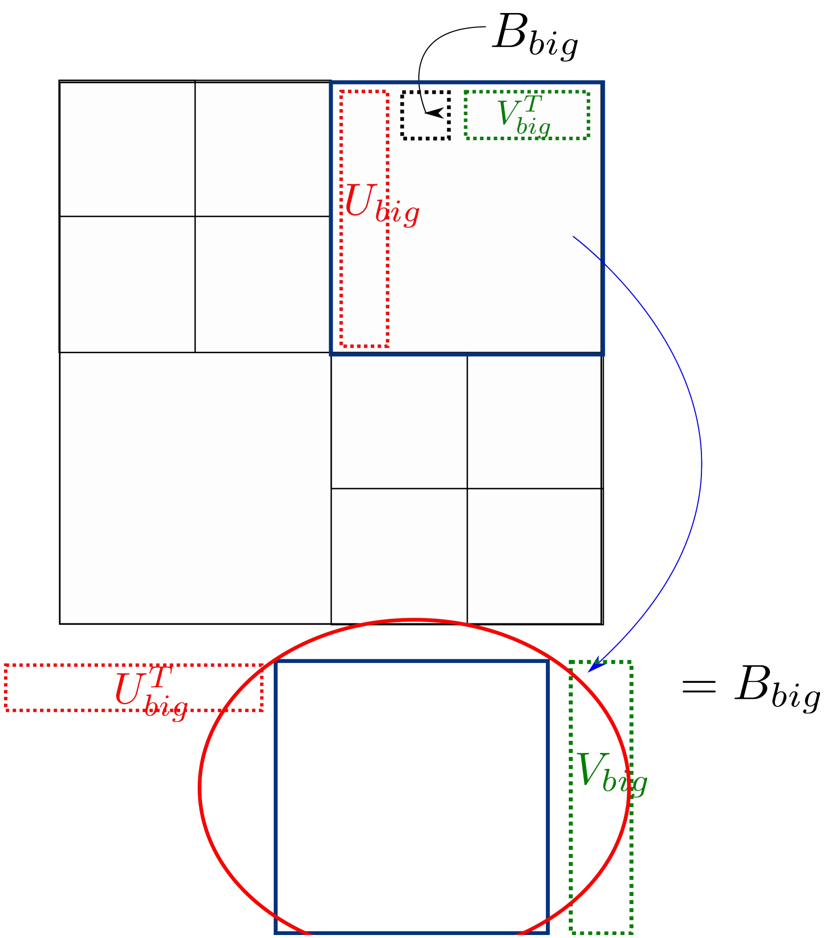

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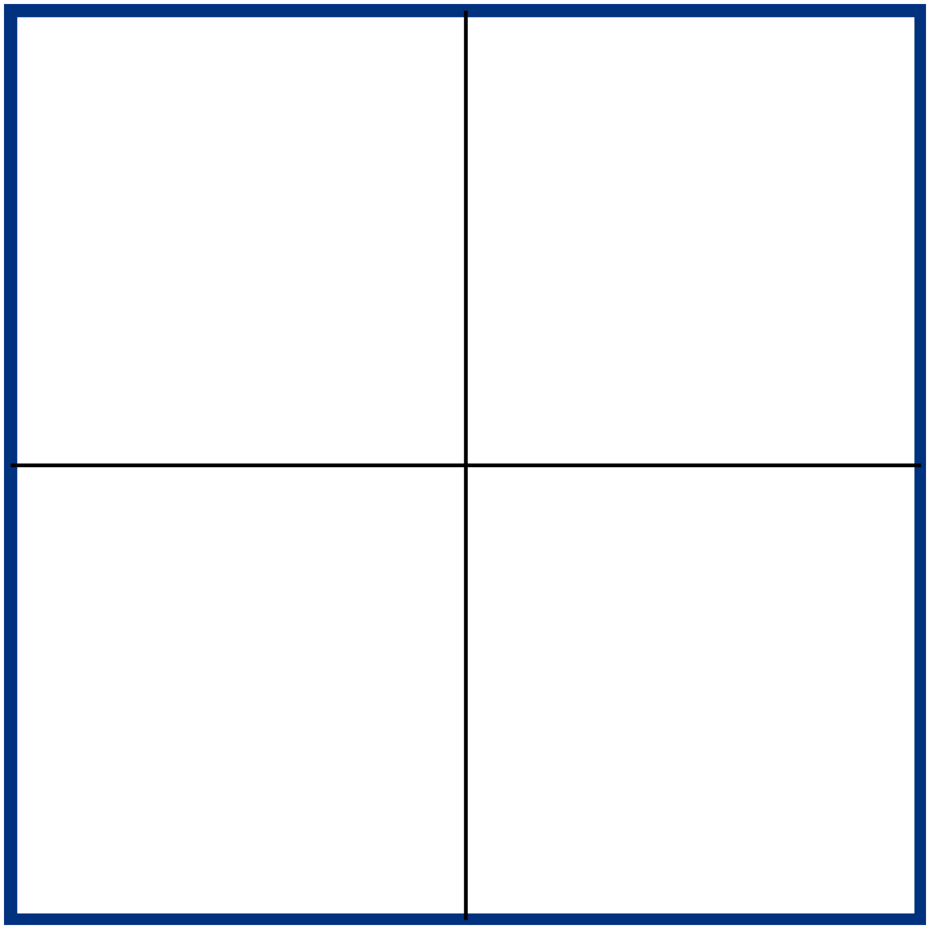
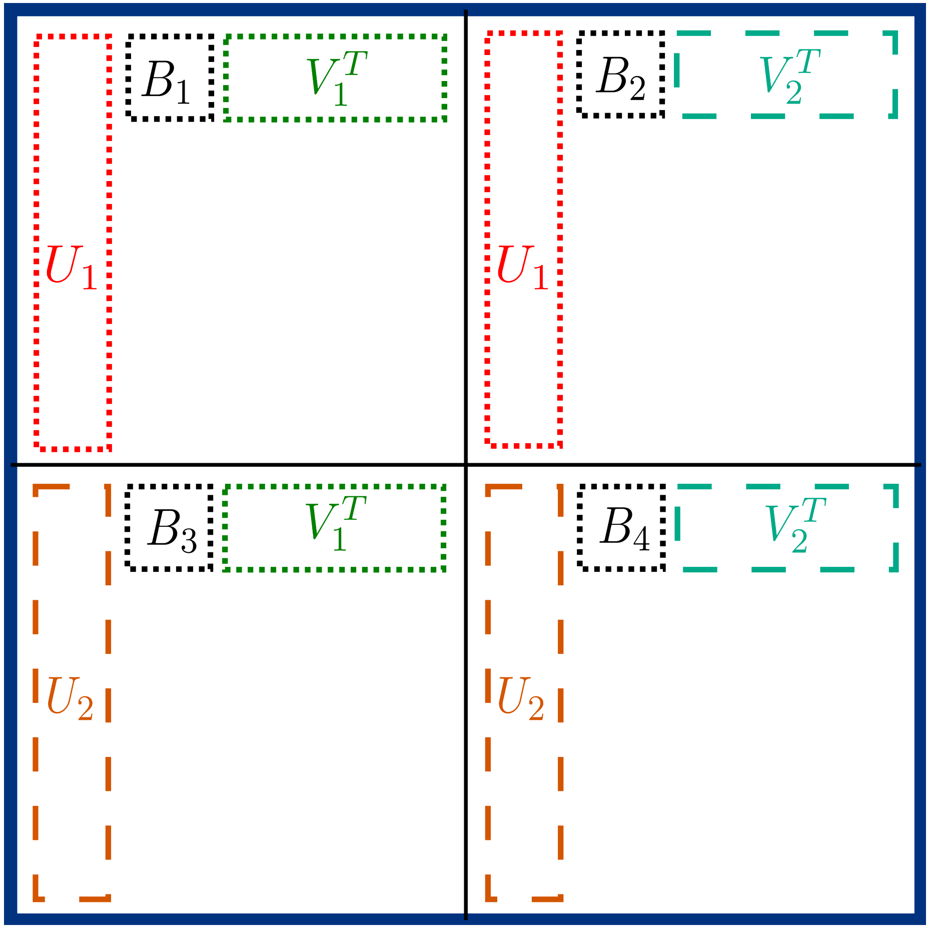


Phase 2 – A Better Way: Divide and Conquer



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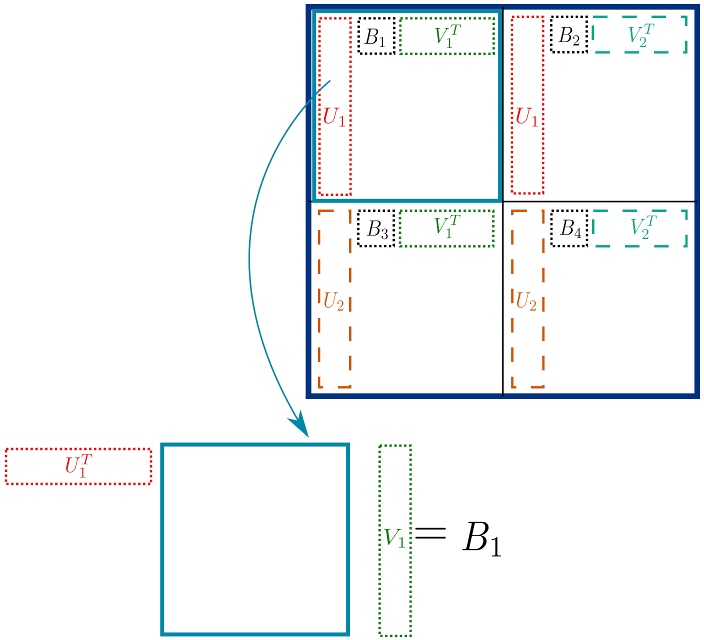

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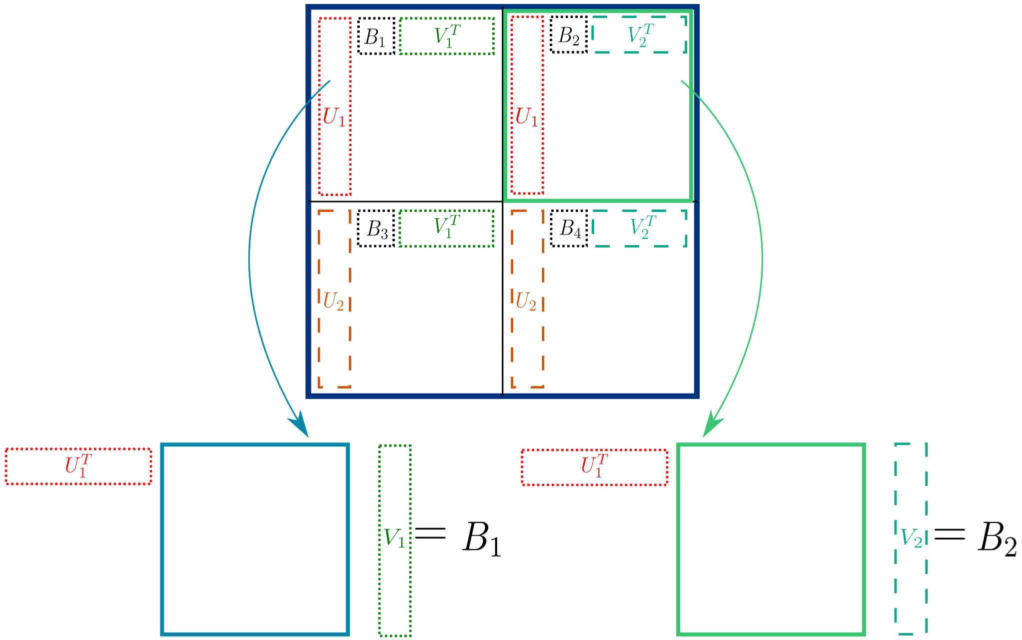

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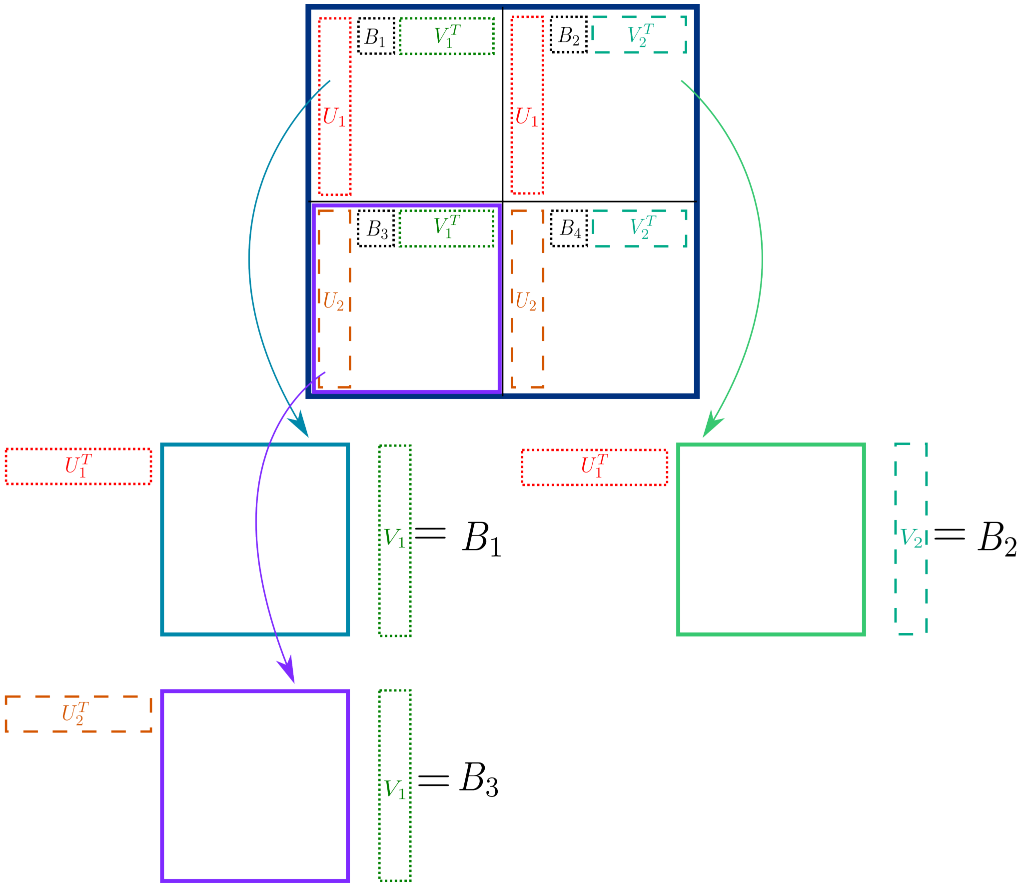

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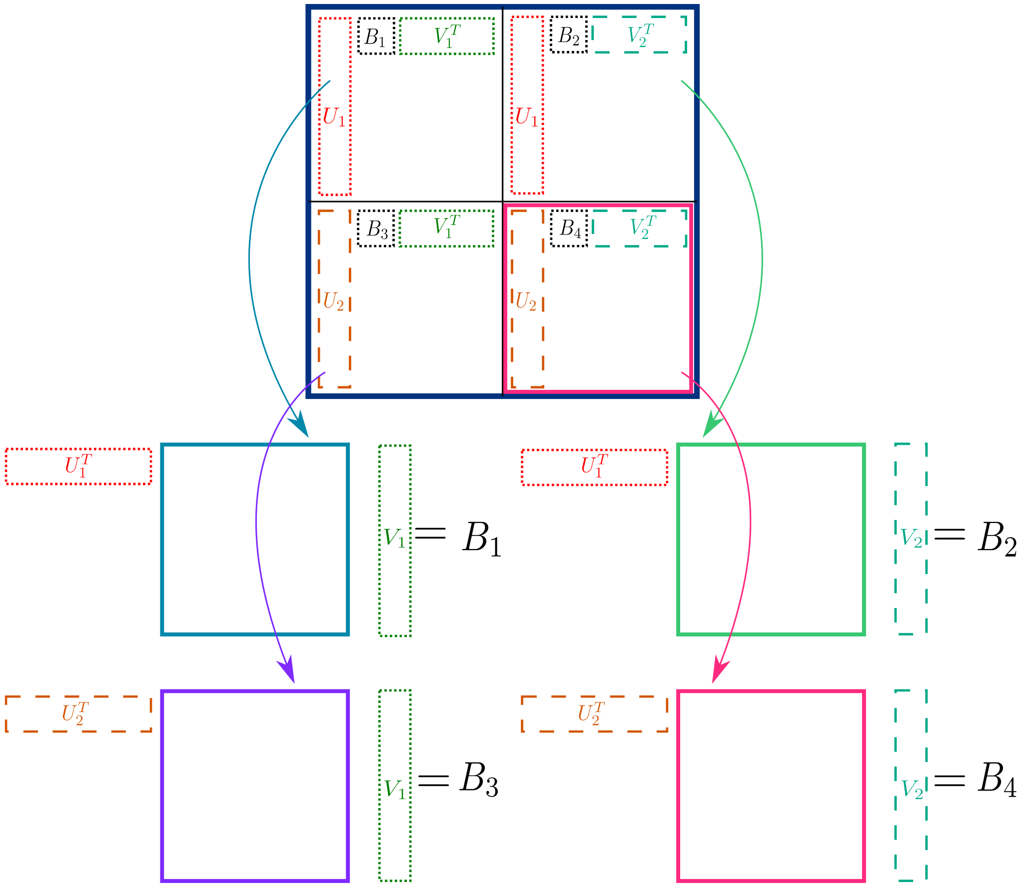

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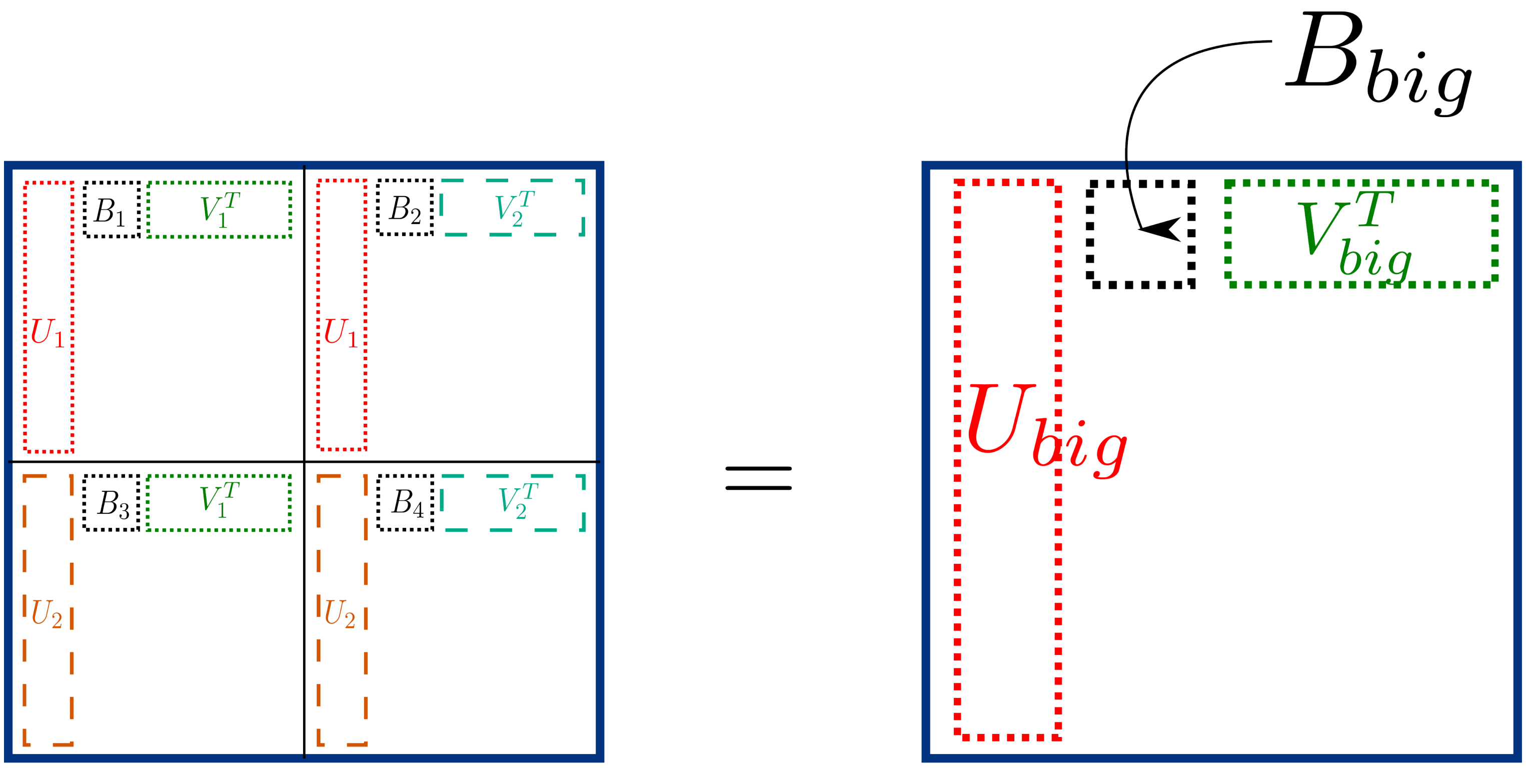

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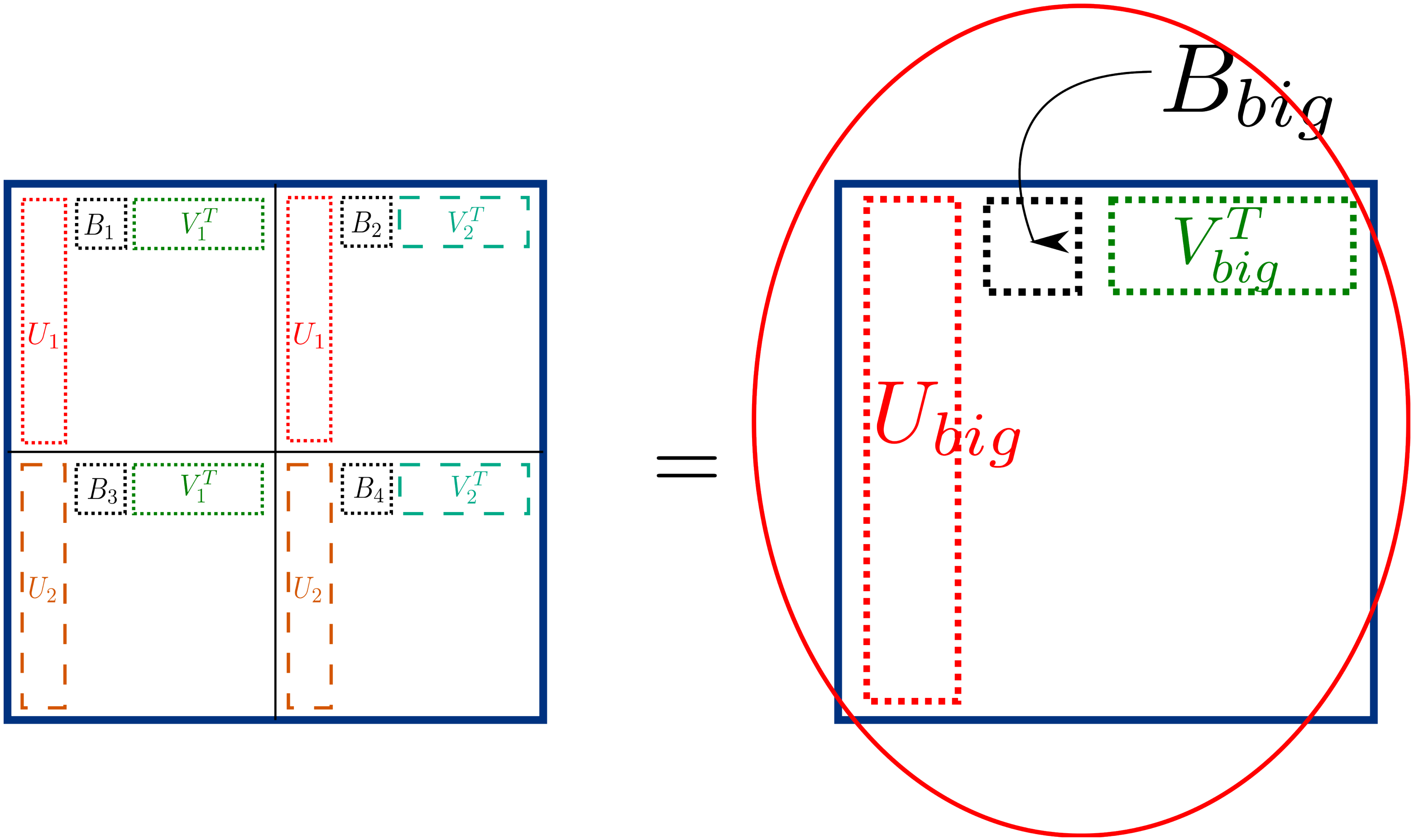

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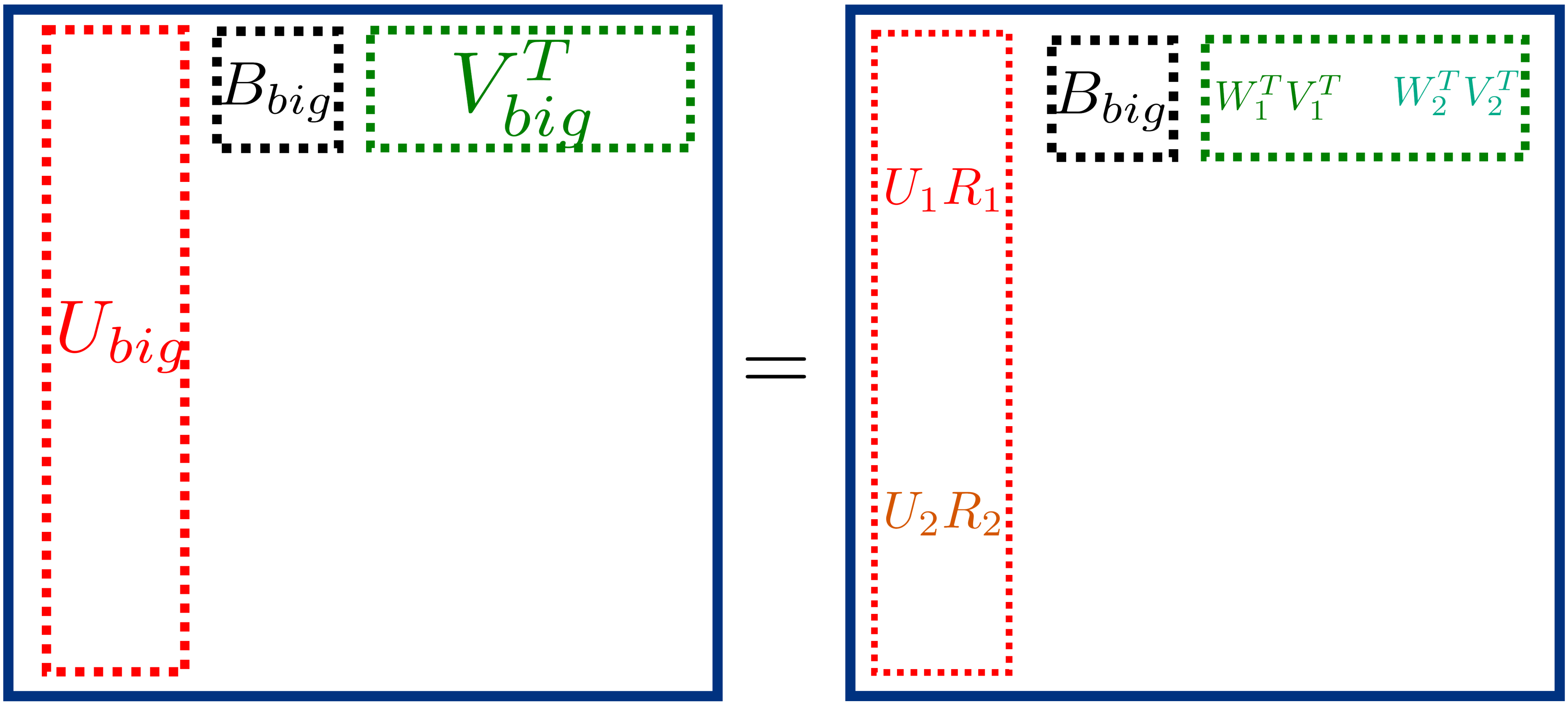

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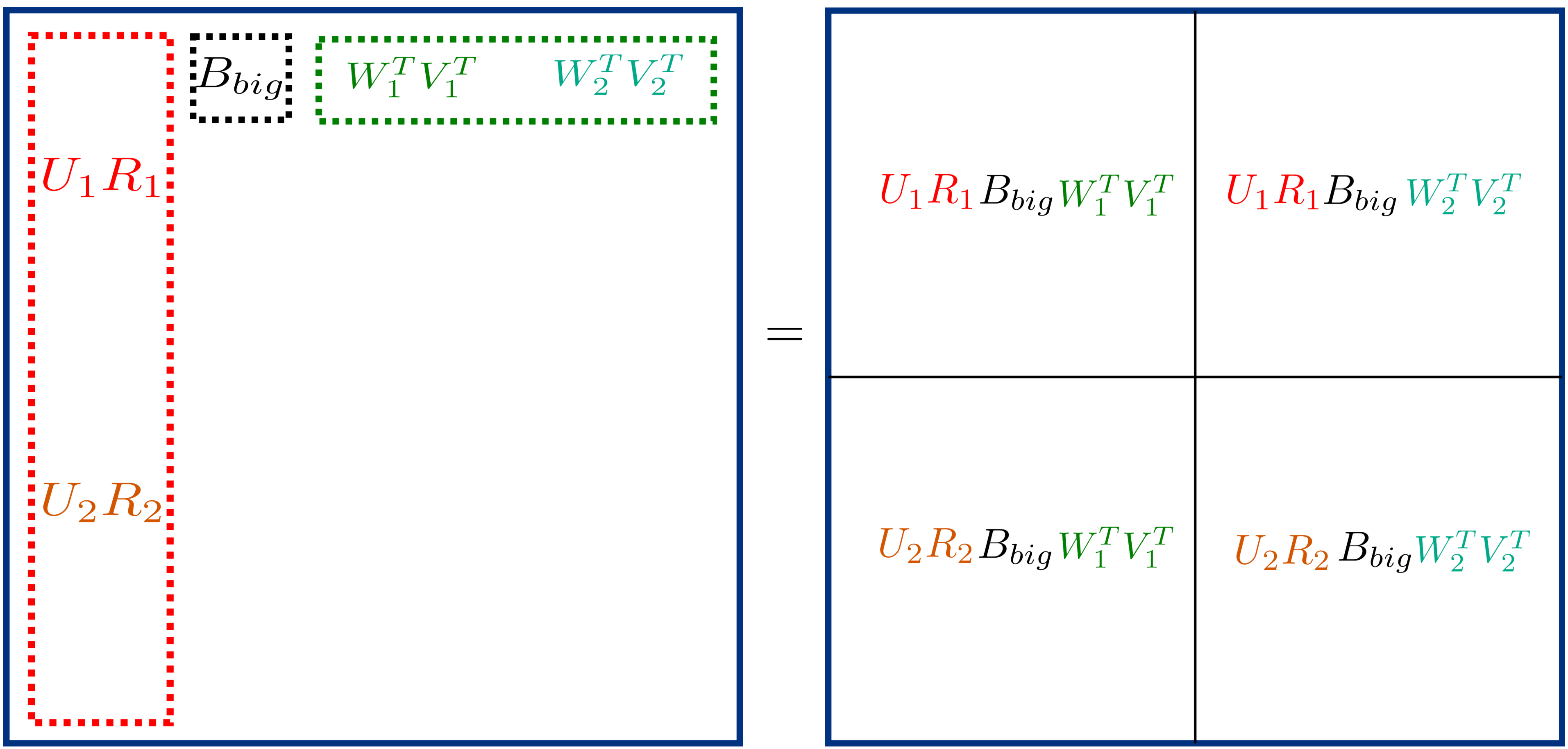
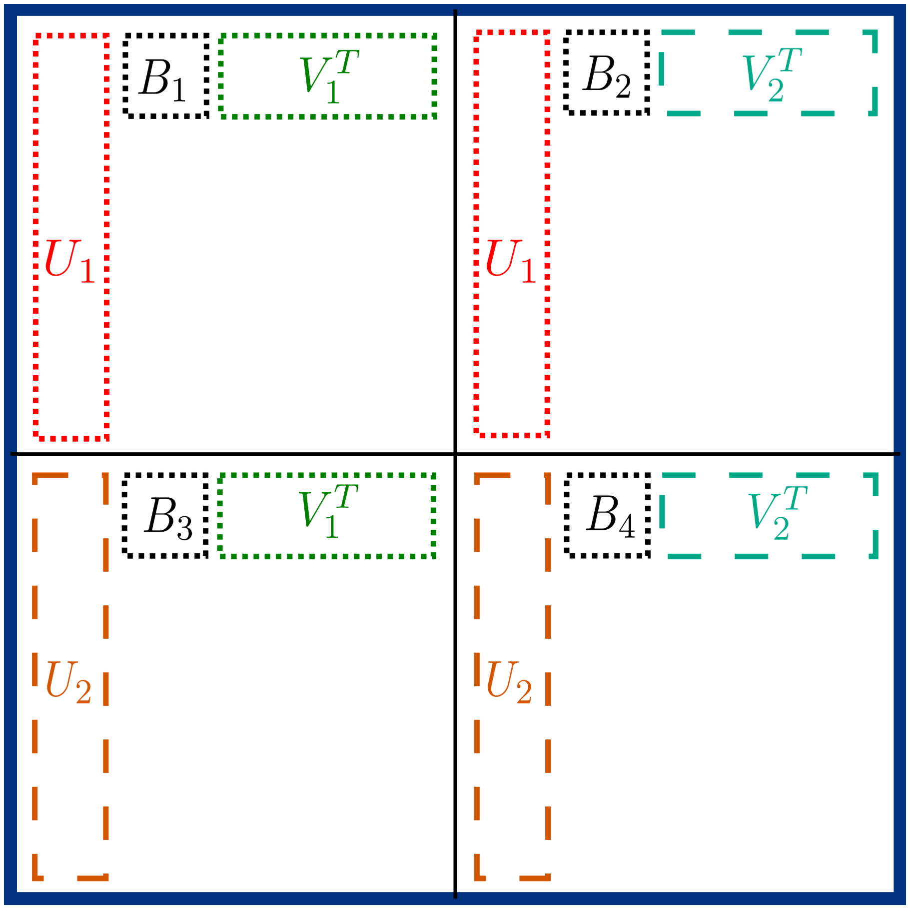


Phase 2 – A Better Way: Divide and Conquer

$$\left[\begin{array}{c|c} U_1 R_1 B_{big} W_1^T V_1^T & U_1 R_1 B_{big} W_2^T V_2^T \\ \hline U_2 R_2 B_{big} W_1^T V_1^T & U_2 R_2 B_{big} W_2^T V_2^T \end{array}\right] =$$



Phase 2 – A Better Way: Divide and Conquer

$$\left[\begin{array}{c|c} U_1 R_1 B_{big} W_1^T V_1^T & U_1 R_1 B_{big} W_2^T V_2^T \\ \hline U_2 R_2 B_{big} W_1^T V_1^T & U_2 R_2 B_{big} W_2^T V_2^T \end{array}\right] = \left[\begin{array}{c|c} U_1 B_1 V_1^T & U_1 B_2 V_2^T \\ \hline U_2 B_3 V_1^T & U_2 B_4 V_2^T \end{array}\right]$$

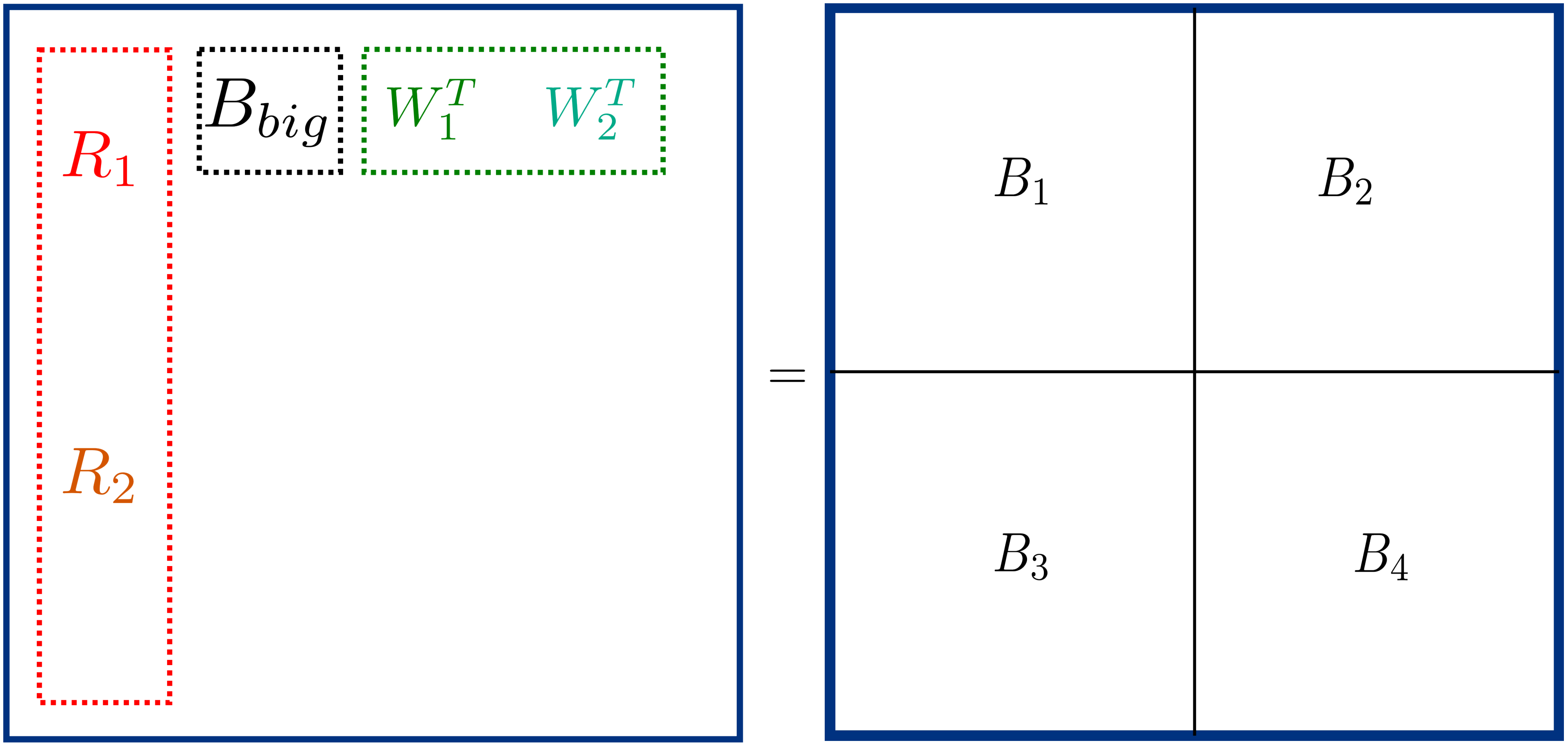
Phase 2 – A Better Way: Divide and Conquer

$$\left[\begin{array}{c|c} R_1 B_{big} W_1^T V_1^T & R_1 B_{big} W_2^T V_2^T \\ \hline R_2 B_{big} W_1^T V_1^T & R_2 B_{big} W_2^T V_2^T \end{array}\right] = \left[\begin{array}{c|c} B_1 V_1^T & B_2 V_2^T \\ \hline B_3 V_1^T & B_4 V_2^T \end{array}\right]$$

Phase 2 – A Better Way: Divide and Conquer

$$\left[\begin{array}{c|c} R_1 B_{big} W_1^T & R_1 B_{big} W_2^T \\ \hline R_2 B_{big} W_1^T & R_2 B_{big} W_2^T \end{array}\right] = \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]$$

Phase 2 – A Better Way: Divide and Conquer



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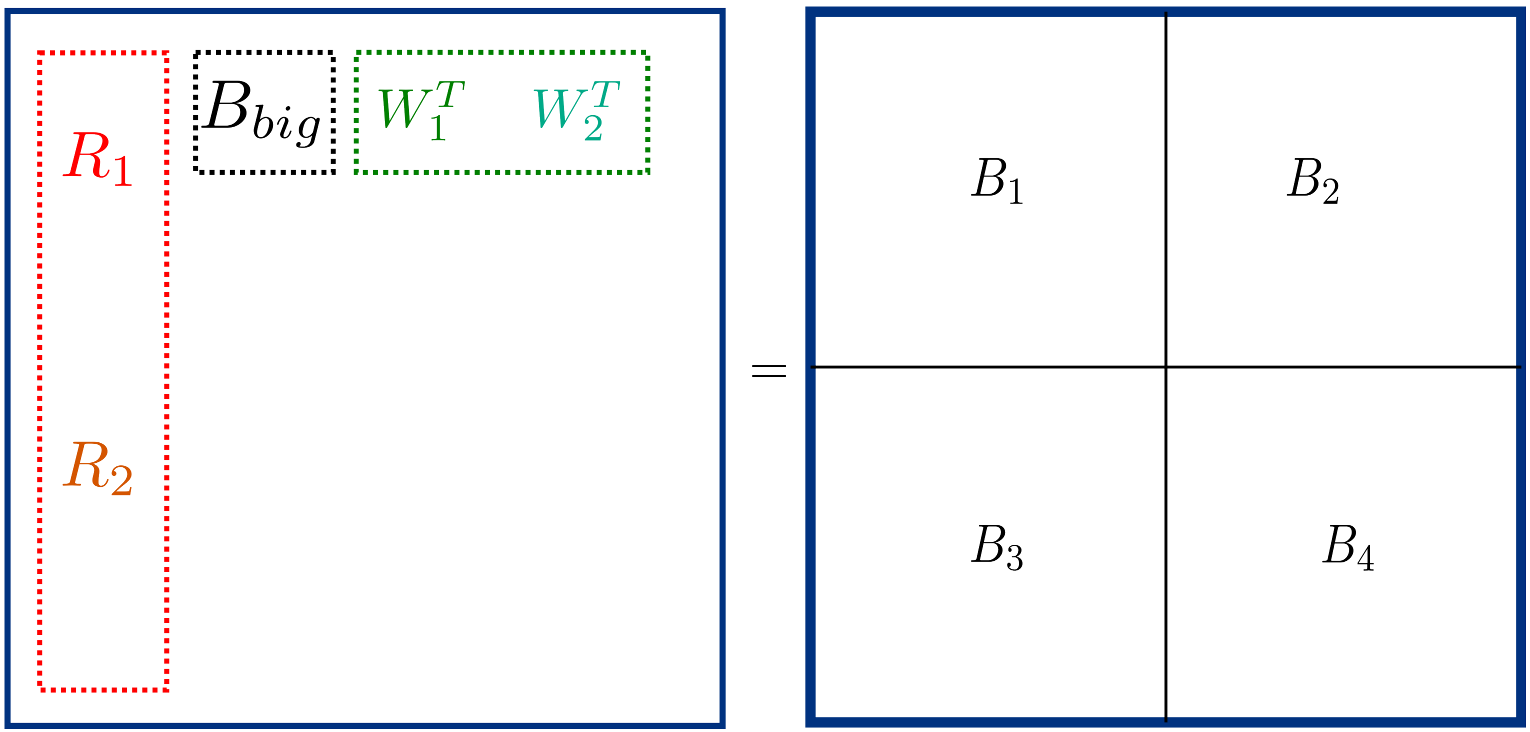


$$\begin{bmatrix} R_1^T & R_2^T \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = I$$

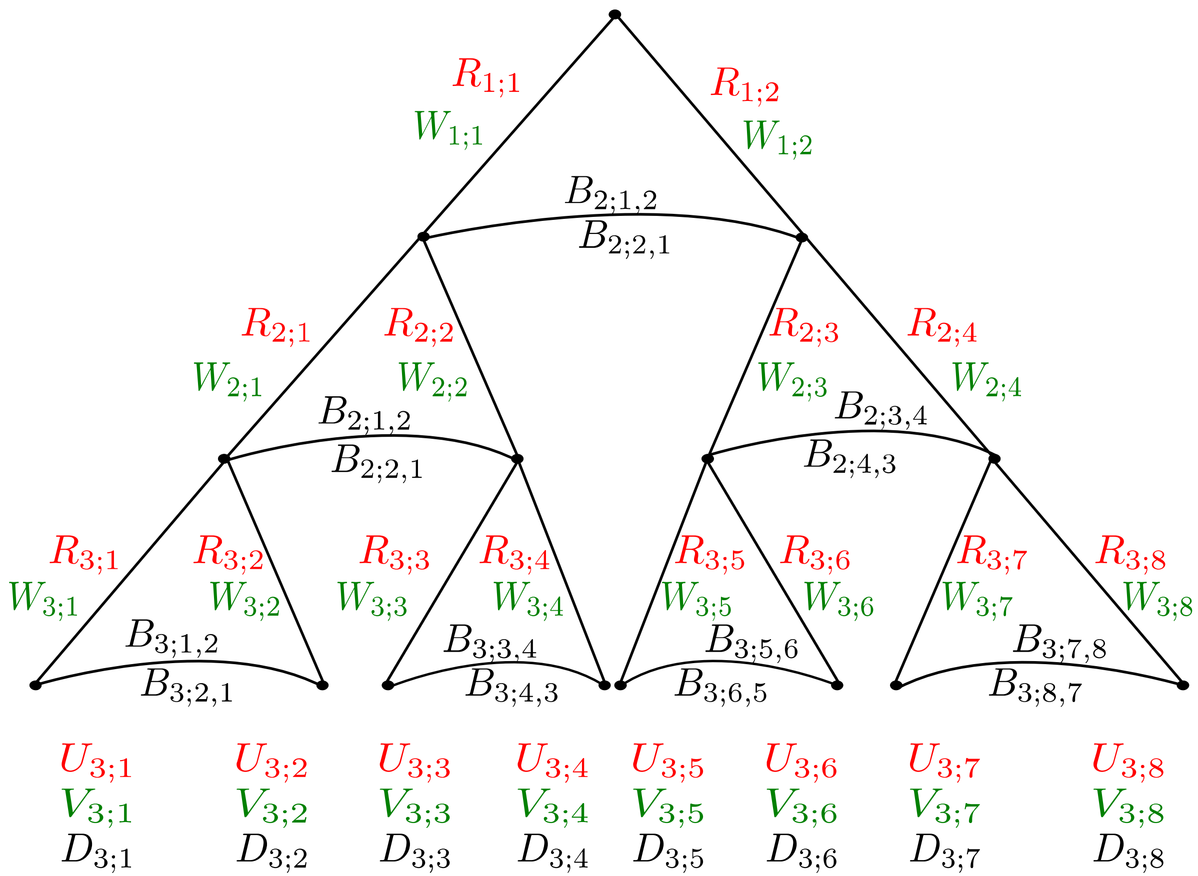
$$\begin{bmatrix} W_1^T & W_2^T \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = I$$

Phase 2 – A Better Way: Divide and Conquer

$$B_{big} = \begin{array}{cc} & \begin{array}{cc} R_1^T & R_2^T \end{array} \\ & \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] \begin{array}{c} W_1 \\ W_2 \end{array} \end{array}$$

Phase 1 & 2 Complete

- We have calculated every $U_{k;i}$,$V_{k;i}$, $R_{k;i}$,$W_{k;i}$ and $B_{k;i,j}$ in the HSS Representation



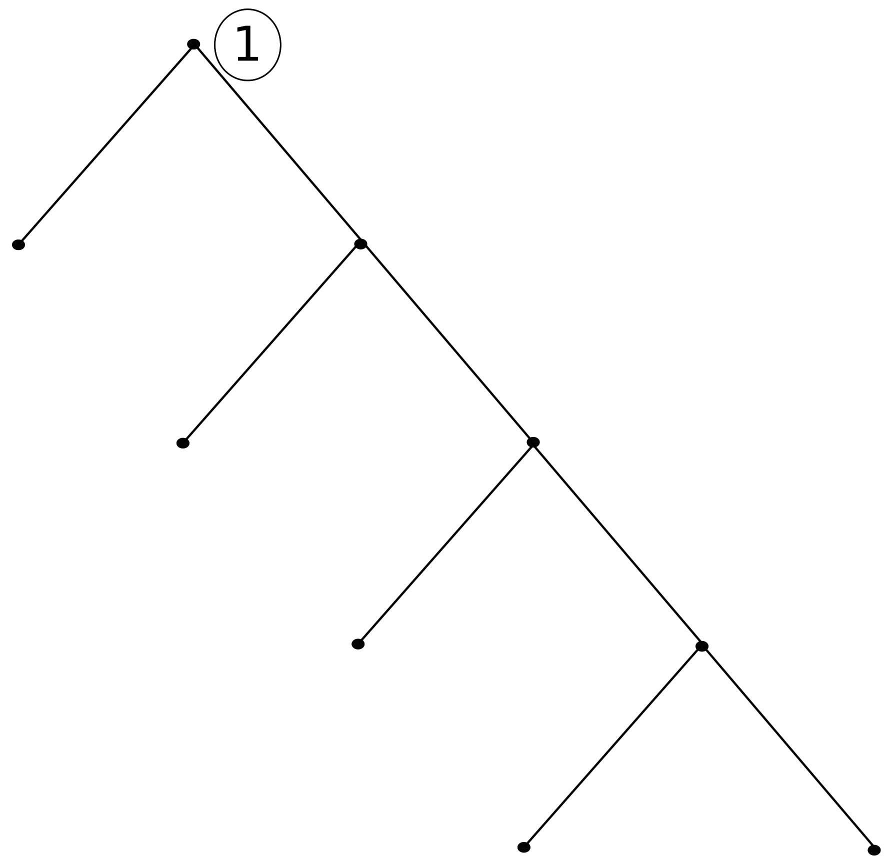
Algorithm Memory Consumption

- Other algorithms can take as much as $O(n^2)$ memory due to a depth first traversal of the HSS tree
- Our algorithm traverses the tree in a deepest first order instead, and takes $O(p^{0.5}n^{1.5})$ memory in the worst case, where p is the rank of the off diagonal blocks of the matrix A, while still taking only $O(n^2p)$ flops

What is the Worst Case Memory Consumption for Our Algorithm?

- Phase 2 of our algorithm (computation of Expansion Coefficients $B_{k;i,j}$) consumes at most $O(pn)$ memory
  - One $p \times p$ block is stored in memory for each recursive call.
  - Tree of max depth is $O(n/p)$
  - This implies $O(pn)$ peak memory consumption for a tree of maximal depth
- We need to focus on Phase 1 (computation of basis matrices, $U_{k;i}$ and $V_{k;i}$, and translation operators $R_{k;i}$ and $W_{k;i}$) of our algorithm in order to determine peak workspace consumption

Depth First Traversal is Not Optimal for Peak Memory Consumption



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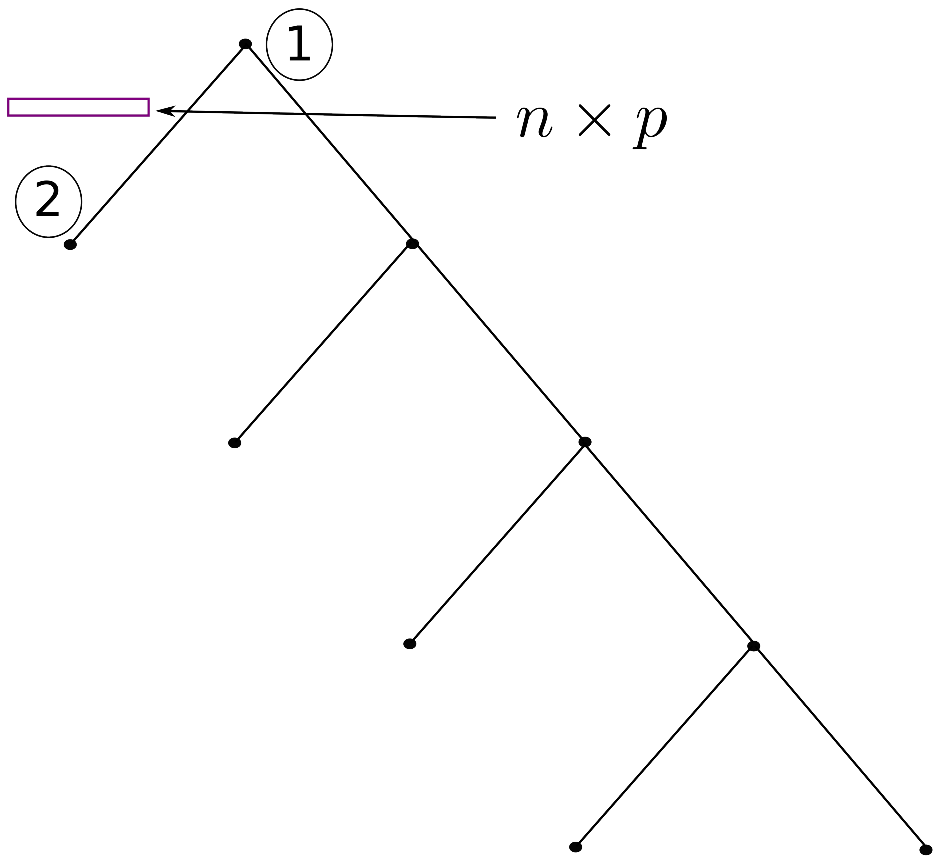
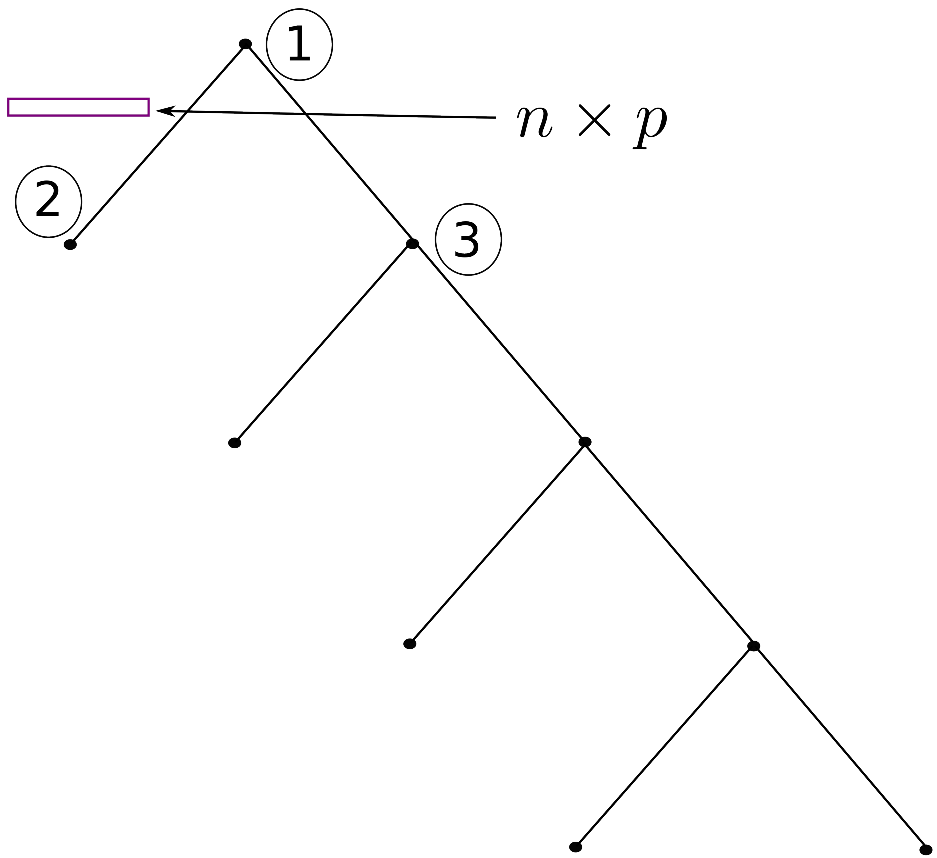


Depth First Traversal is Not Optimal for Peak Memory Consumption



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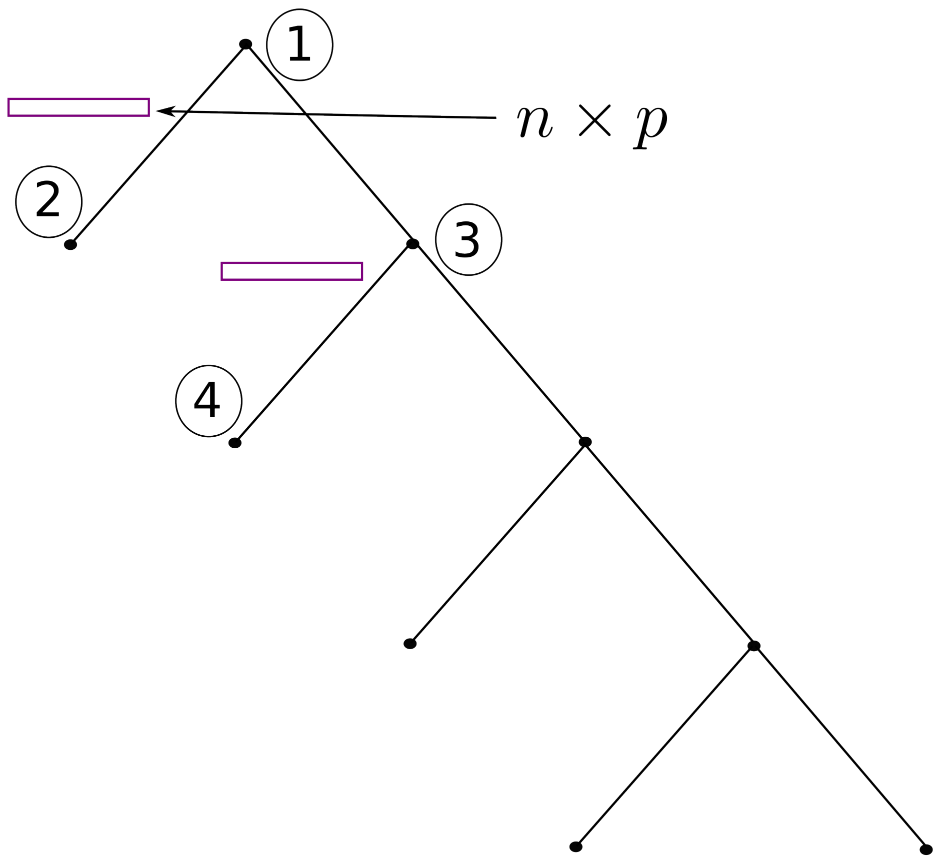

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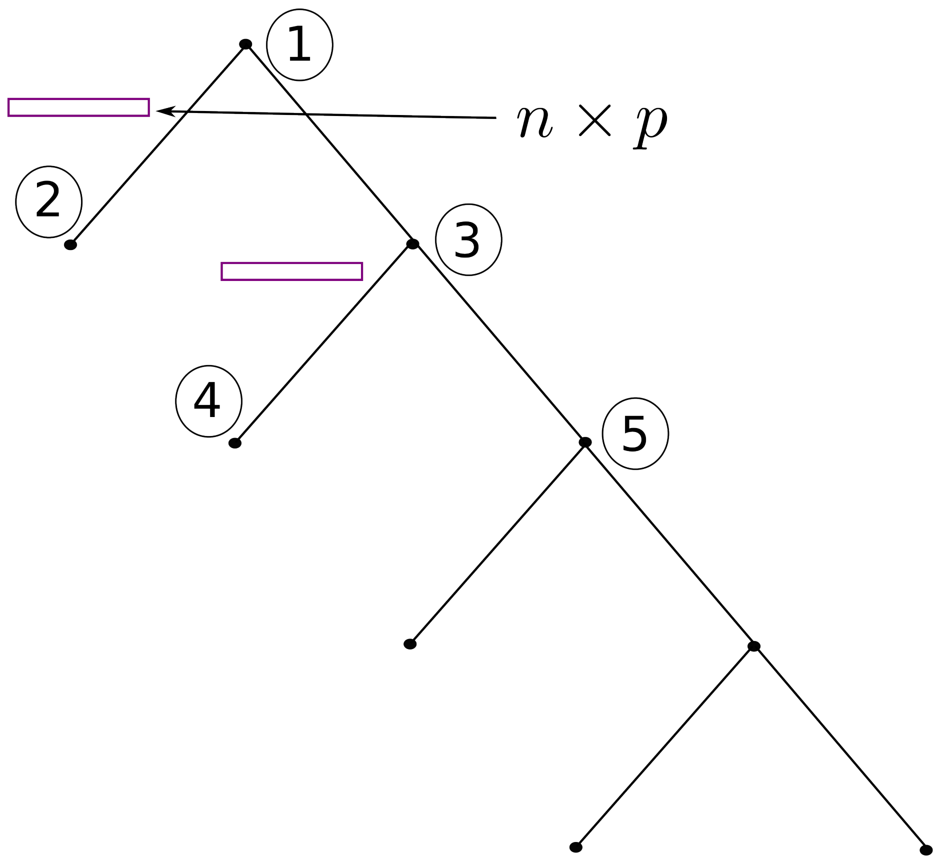

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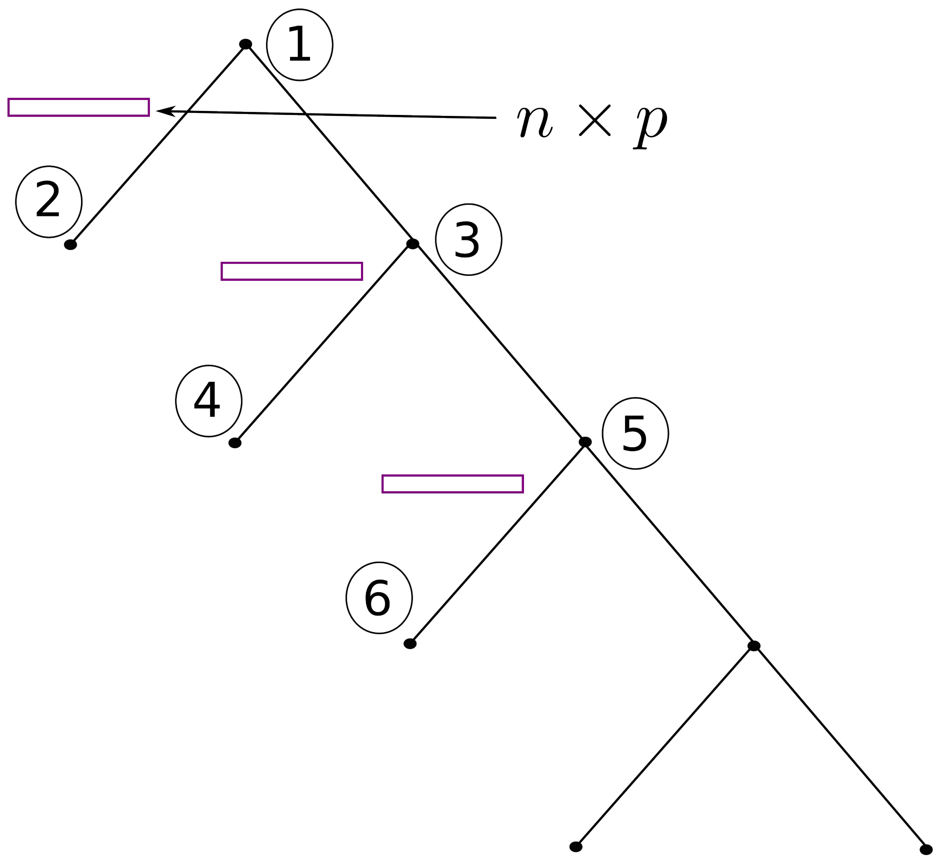

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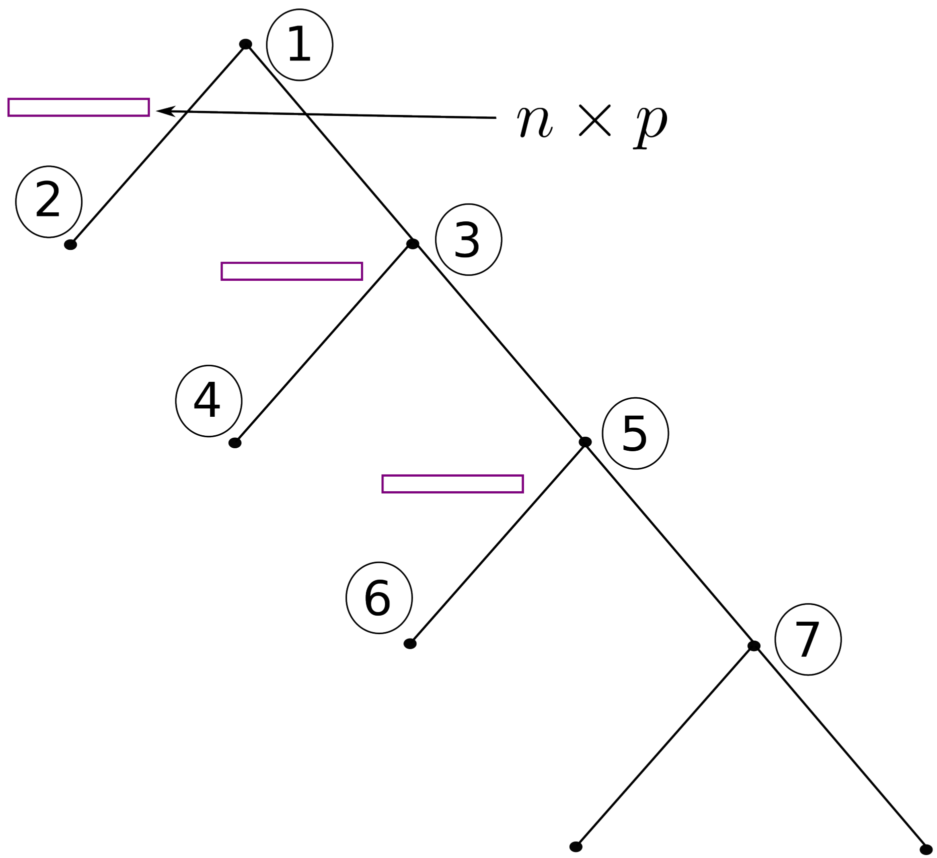
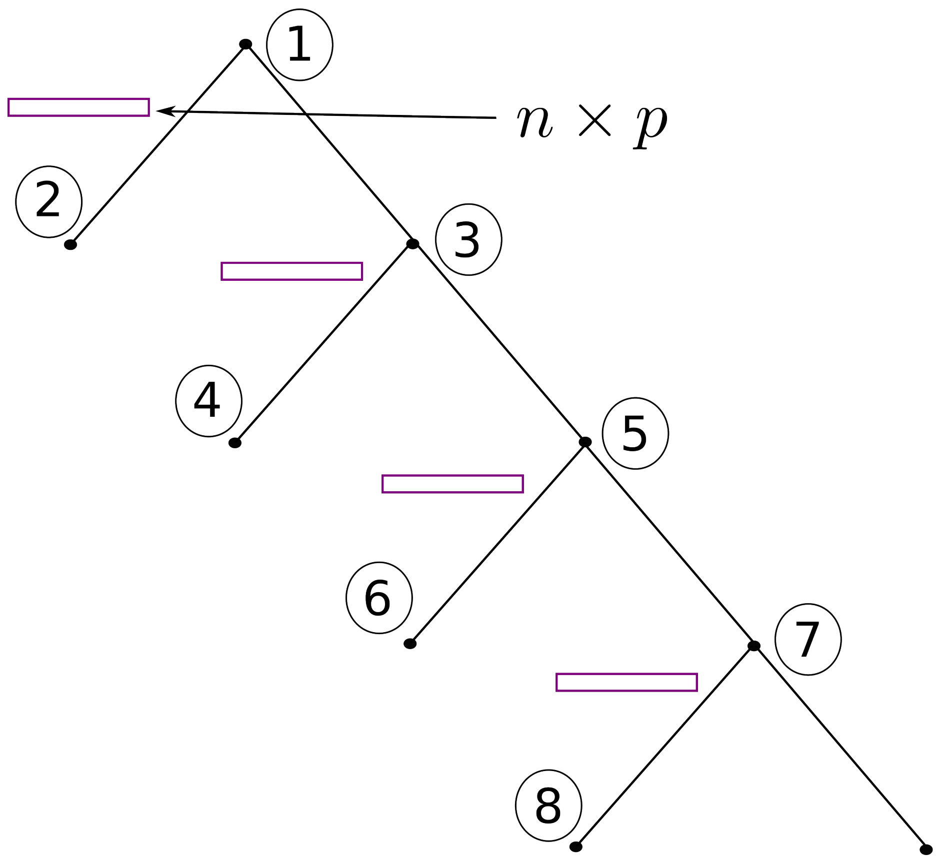


Depth First Traversal is Not Optimal for Peak Memory Consumption



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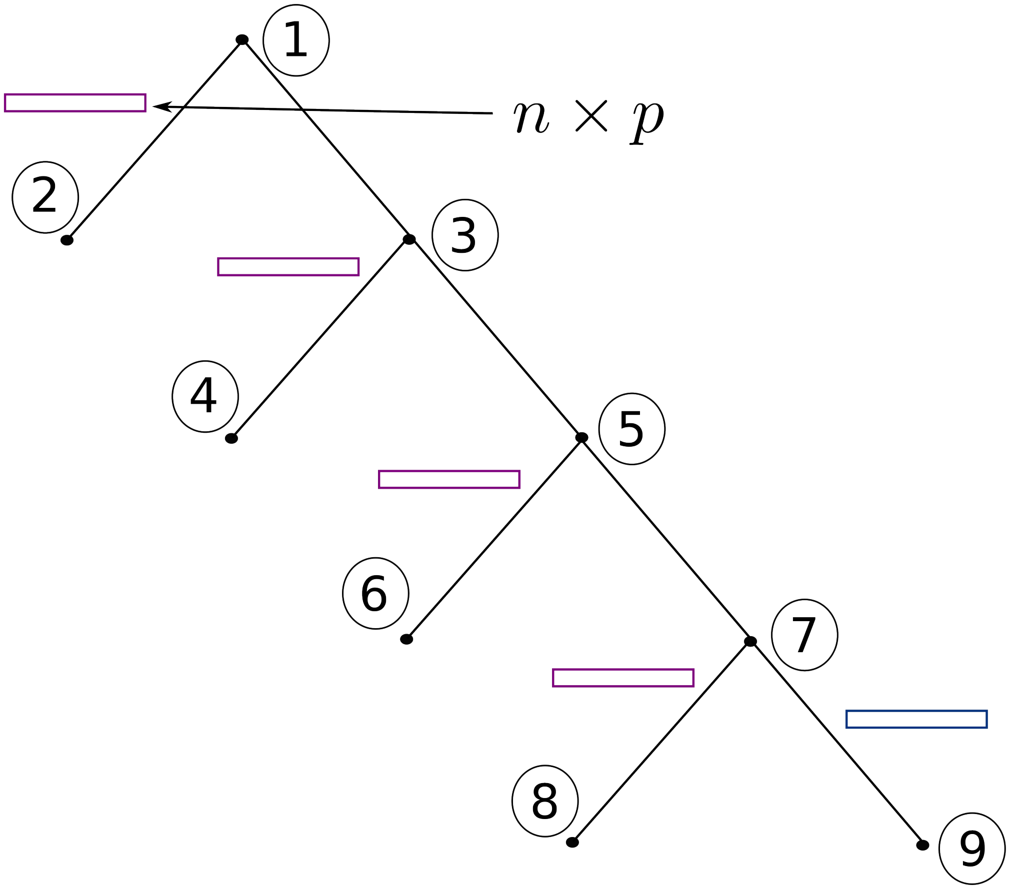


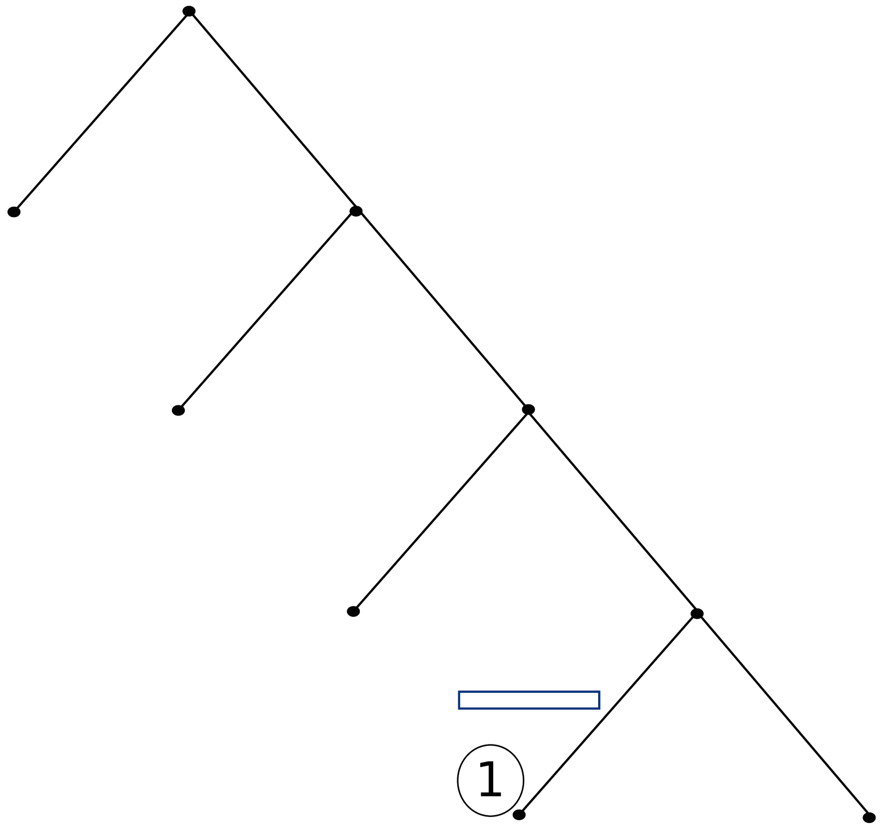
Depth First Traversal is Not Optimal for Peak Memory Consumption

- Each block shown is of dimension $n \times p$, where p is the rank of the off-diagonal blocks of the original matrix
- Maximum Depth of this tree is $O(n/p)$
- This implies a memory consumption of $O(n^2)$

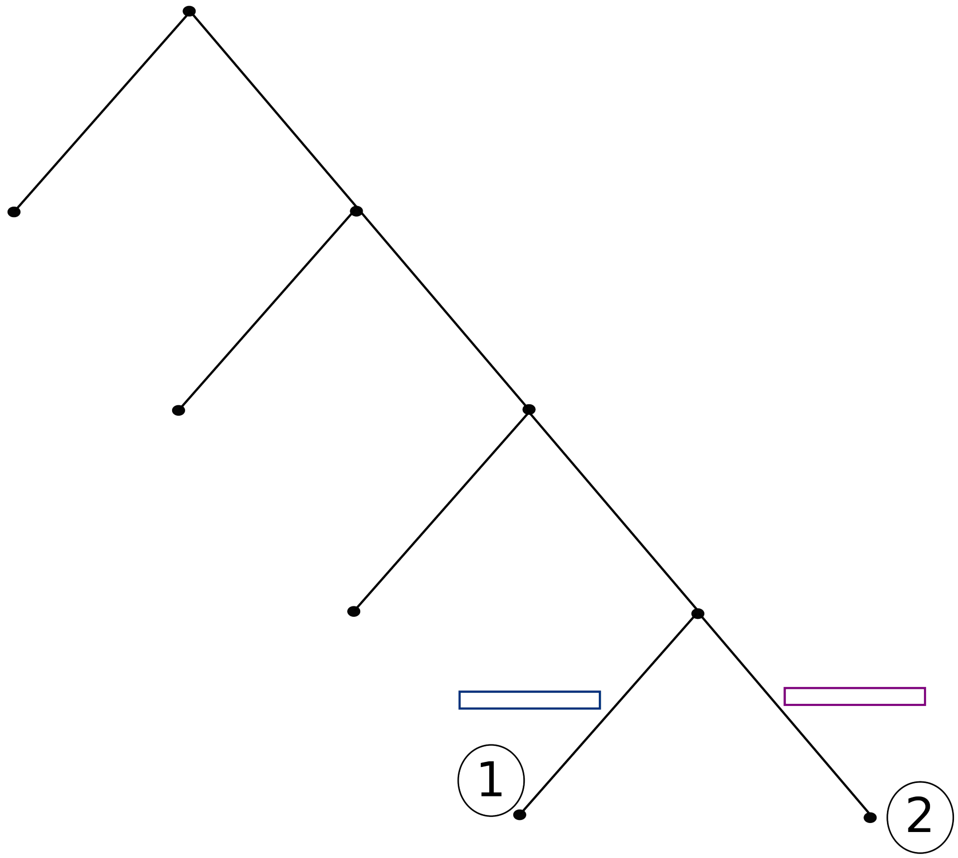
How to fix this: Deepest First Traversal

- Traverse in a Deepest first ordering
- Peak workspace memory consumption of $O(np)$ for a tree of maximal depth

Method We Use: Deepest First Traversal



Method We Use: Deepest First Traversal



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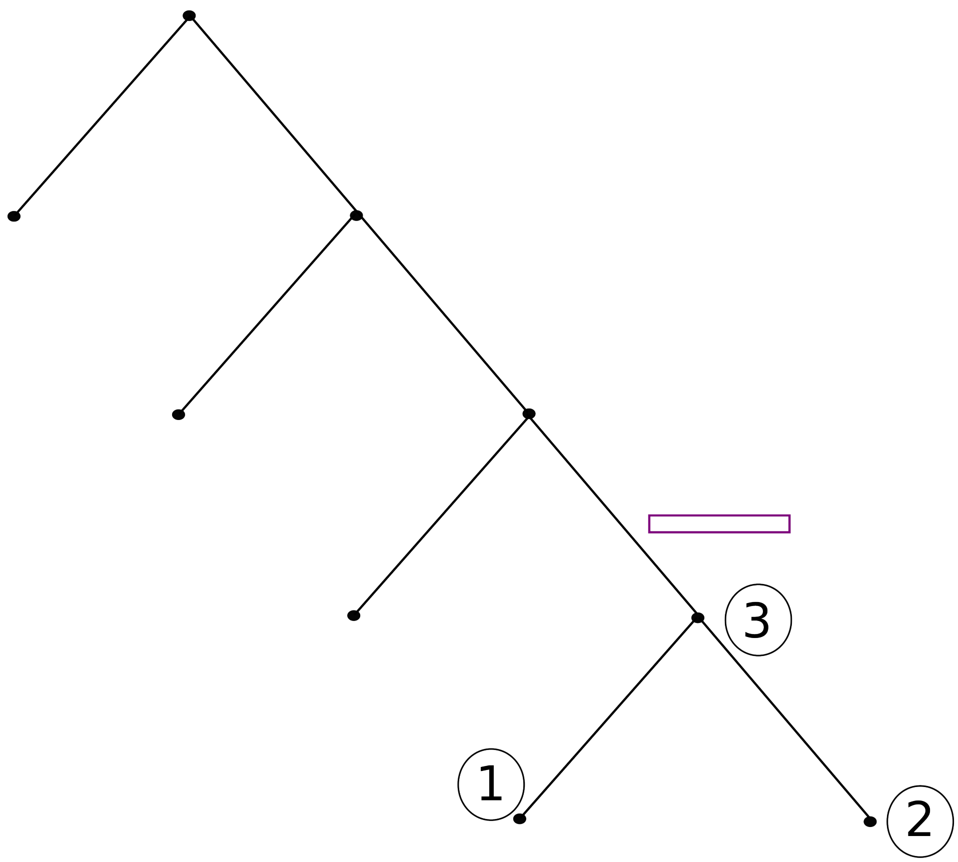

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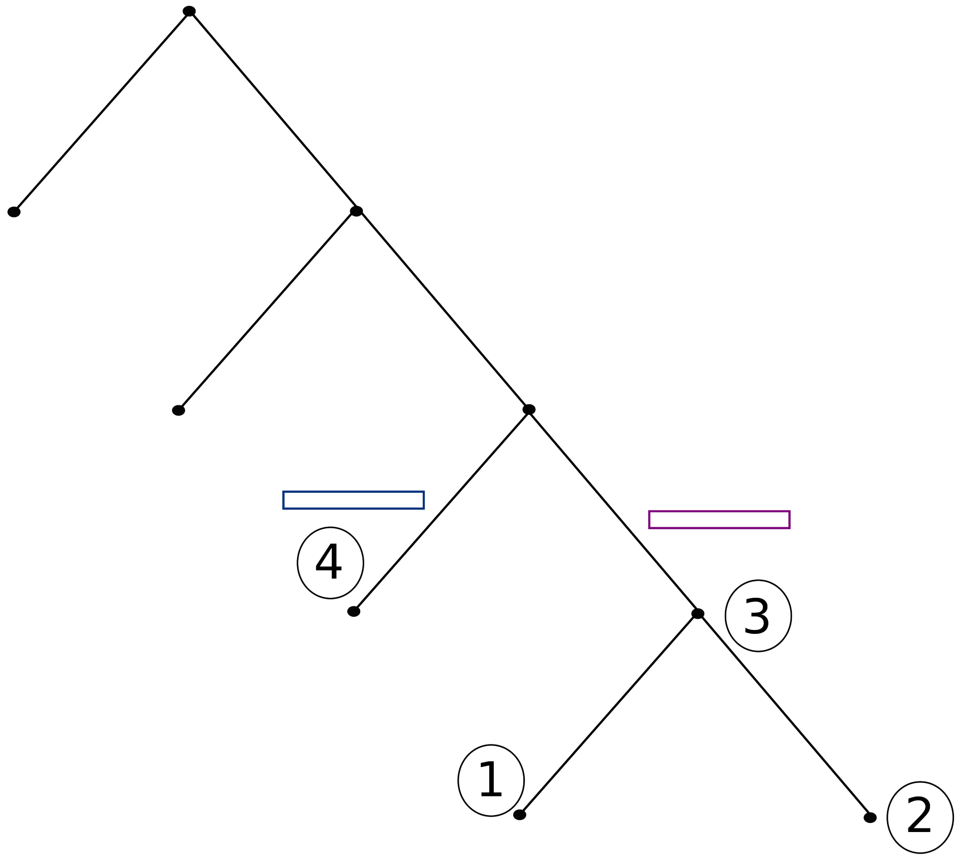

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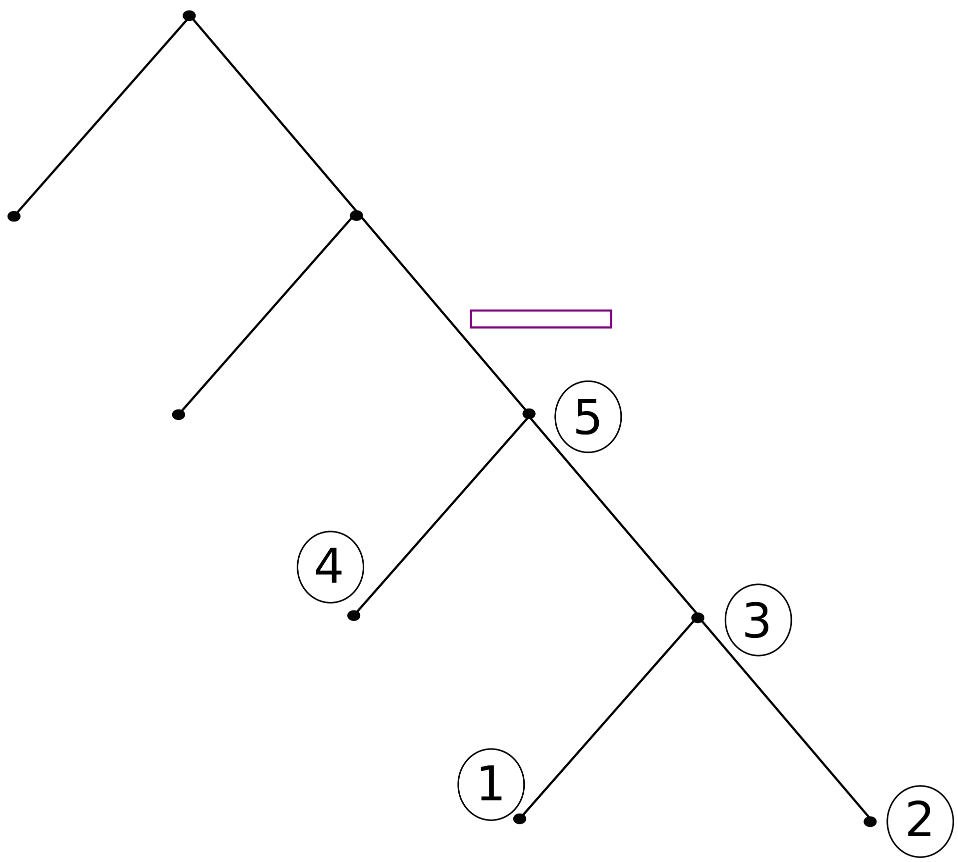

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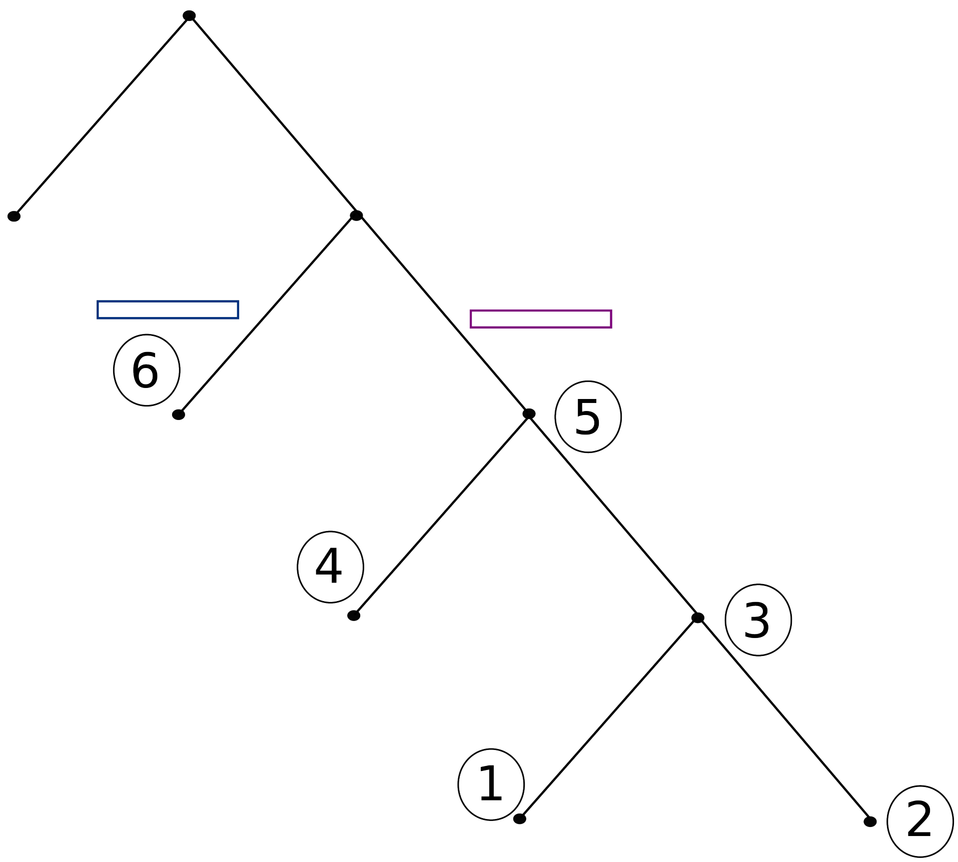

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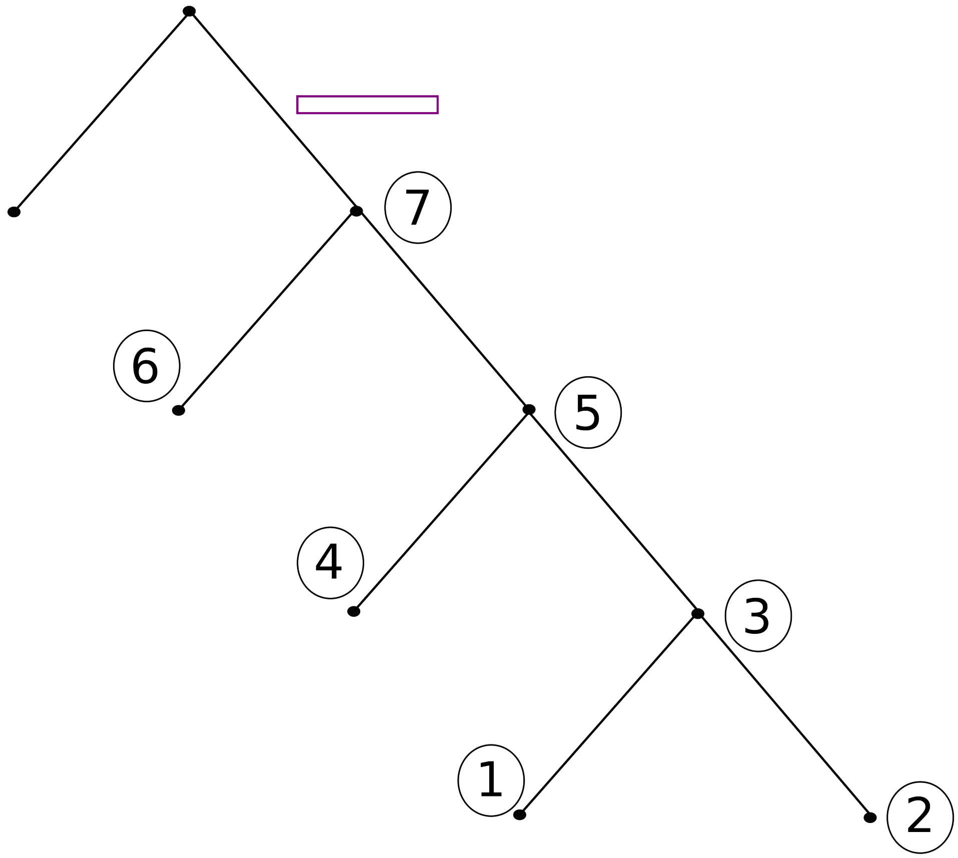

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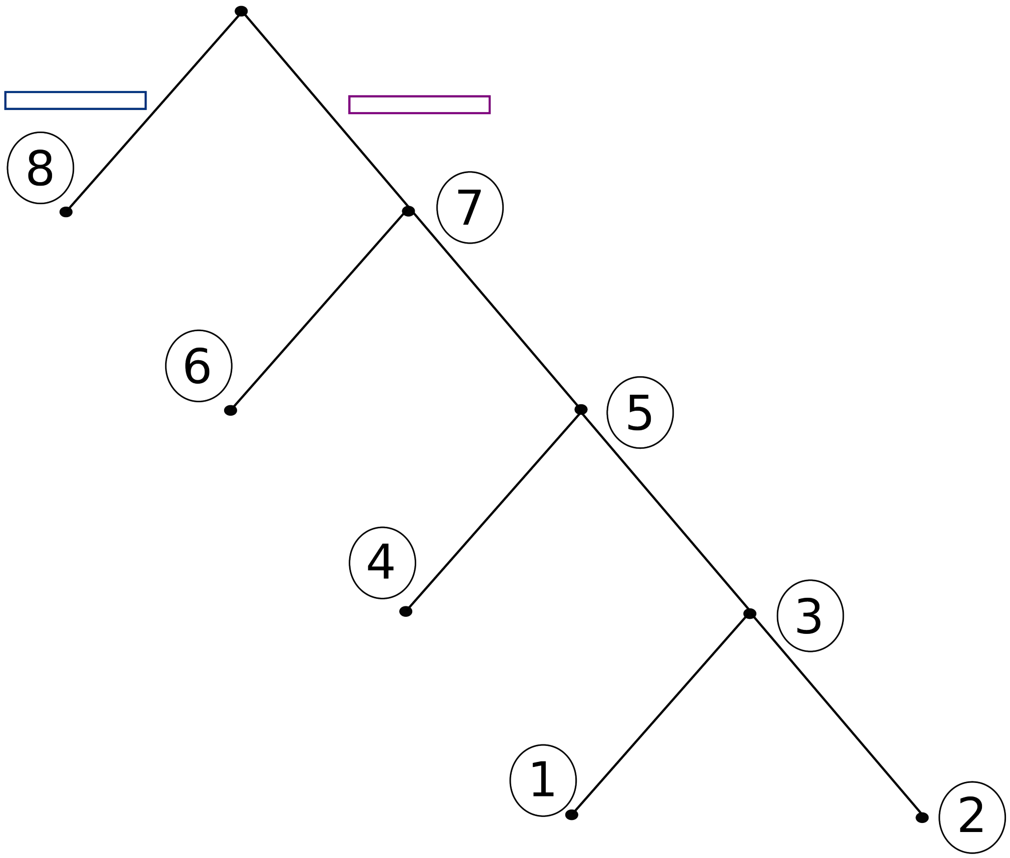

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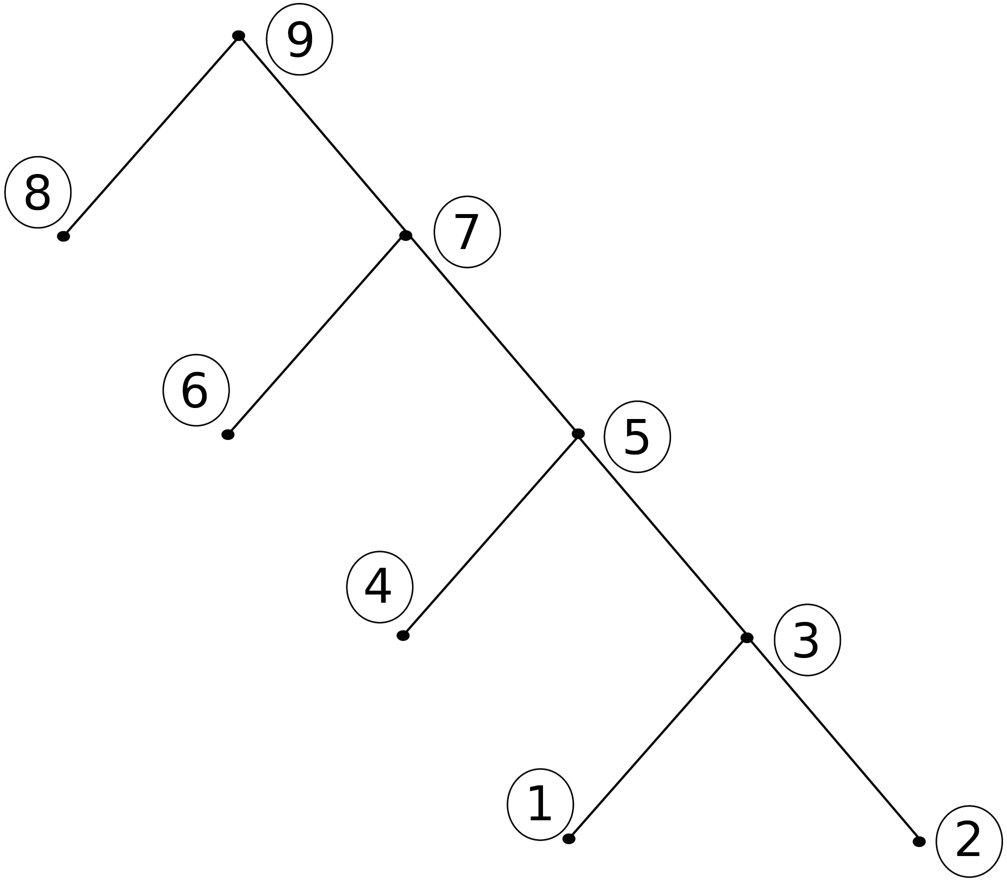


Deepest First Traversal Memory Count

- For the maximal depth tree, only 2 blocks of size $n \times p$ are in memory at any given time
- Peak workspace consumption for a tree of maximal depth is $O(pn)$ using deepest first traversal vs $O(n^2)$ for depth-first traversal
- Further, for a complete tree, the deepest first traversal leads to a peak workspace consumption of $O(pn \log n)$

Worst Case Memory Consumption for our Algorithm

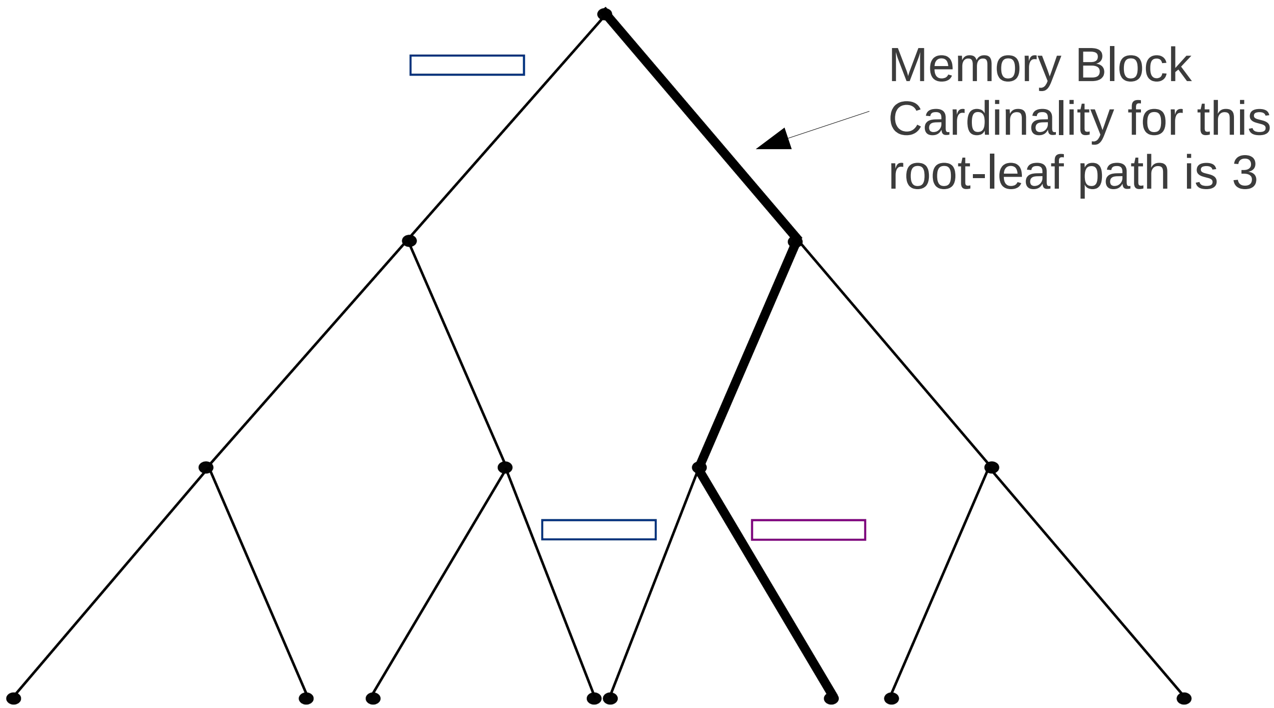
- How does worst case memory usage grow with matrix size, n?
- This can be formulated as a graph theory problem

Memory Block Cardinality for a Root-leaf Path

- Without loss of generality, any HSS tree can be re-ordered such that the depth of the left subtree is always equal to or greater than the depth of the right subtree.

- Block is stored in memory when we return from a left call.

- Memory Block Cardinality for a root-leaf path is equal to the number of right children in that path plus one.



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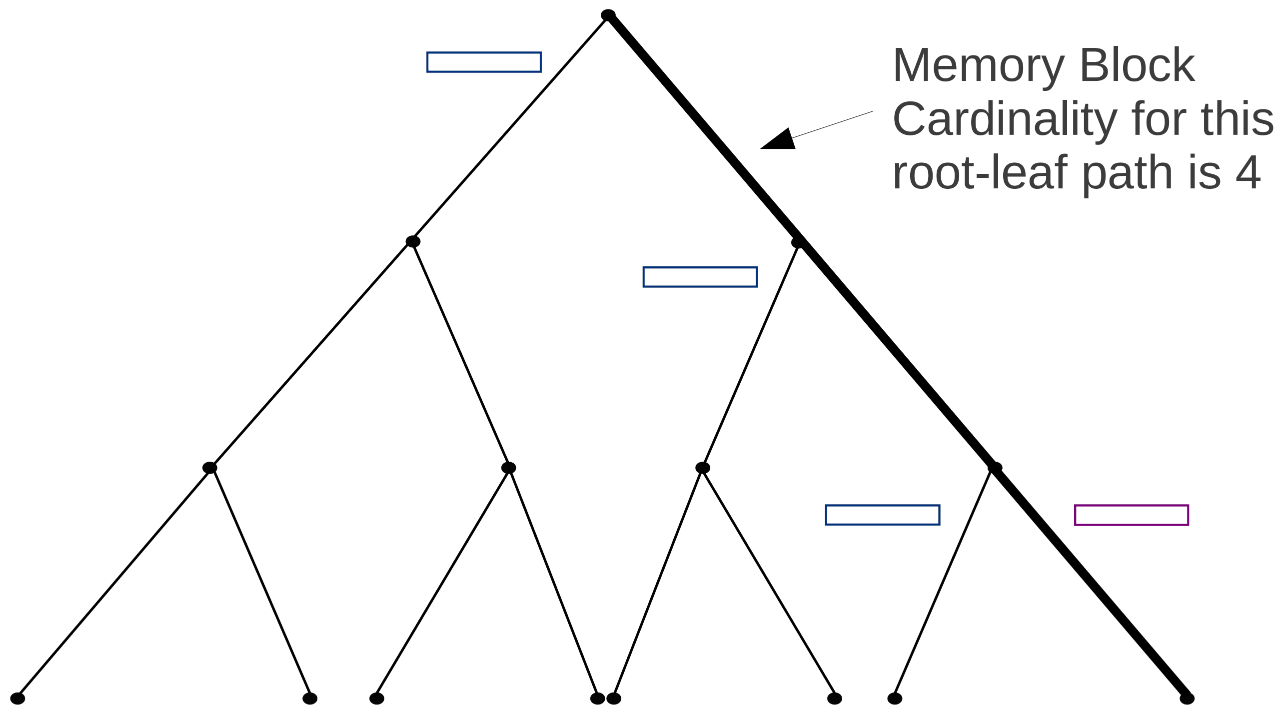
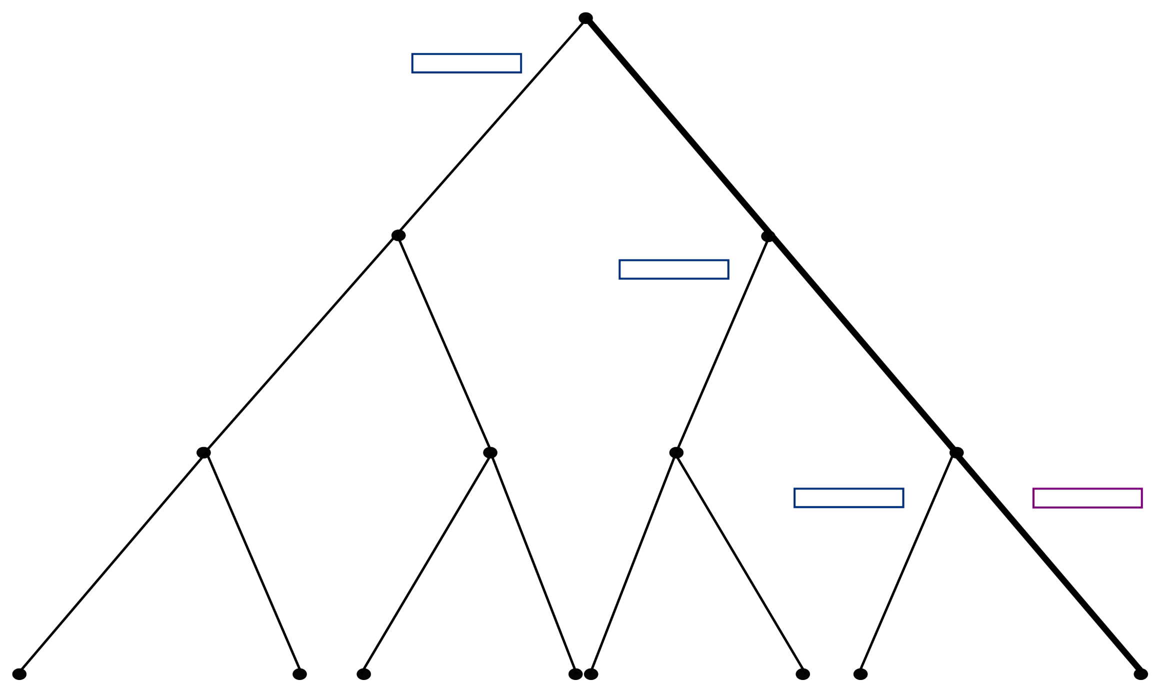


The Search for Maximum Memory Consumption Can Be Formulated as a Graph Theory Problem

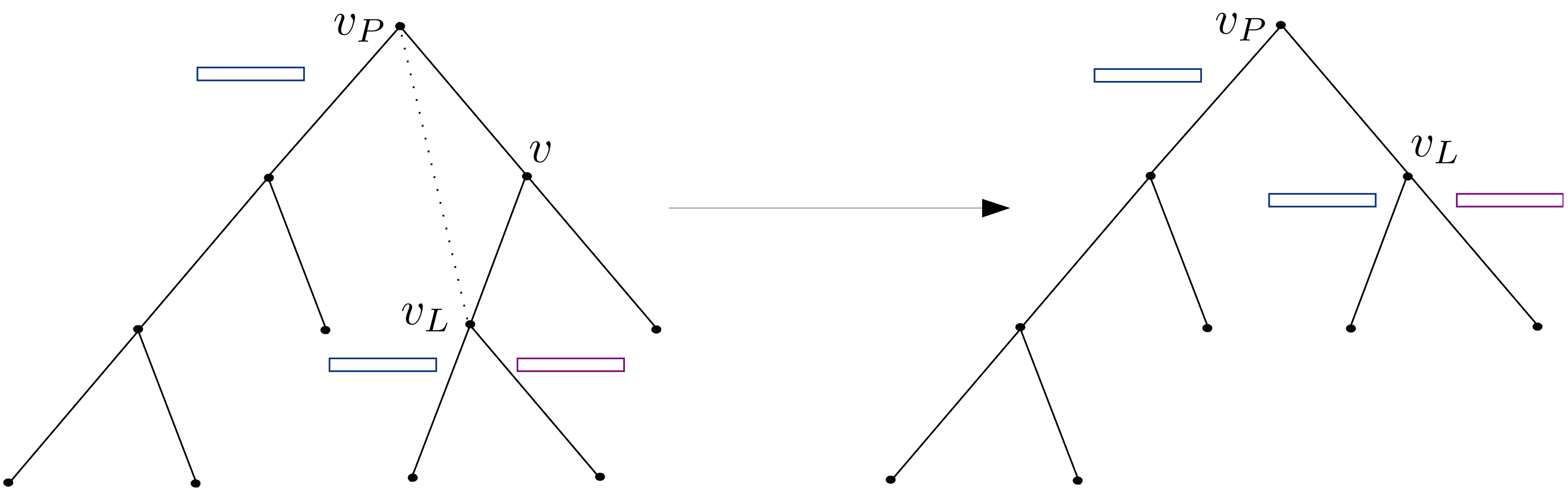
- Branch with maximum memory block cardinality will give peak memory consumption.
- Number of leaf nodes, N_L , is proportional to the size of the matrix n.
- Number of leaf nodes, N_L, is proportional to the number of nodes, N.
- We are looking for a class of trees that maximizes the ratio of the worst case memory block cardinality to number of nodes.



We Can Narrow Down Our Search By Excluding Some Classes of Trees

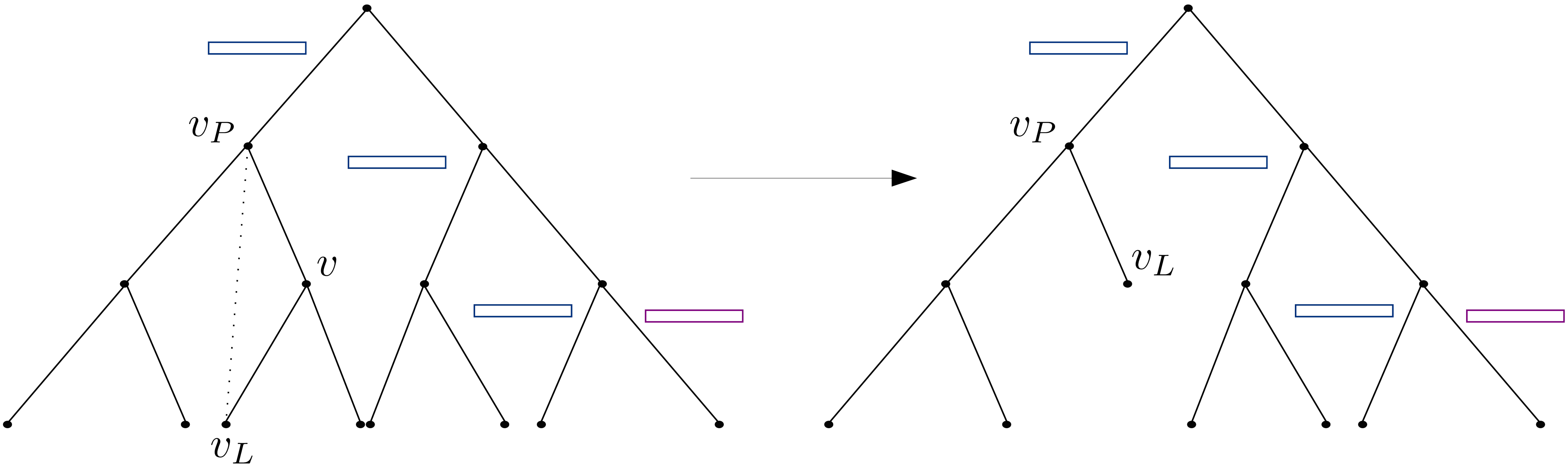
- We can rule out the class of trees that don't have the worst case memory block cardinality along their right-most branch.

- Worst-case memory block cardinality = 3 for both trees shown below.

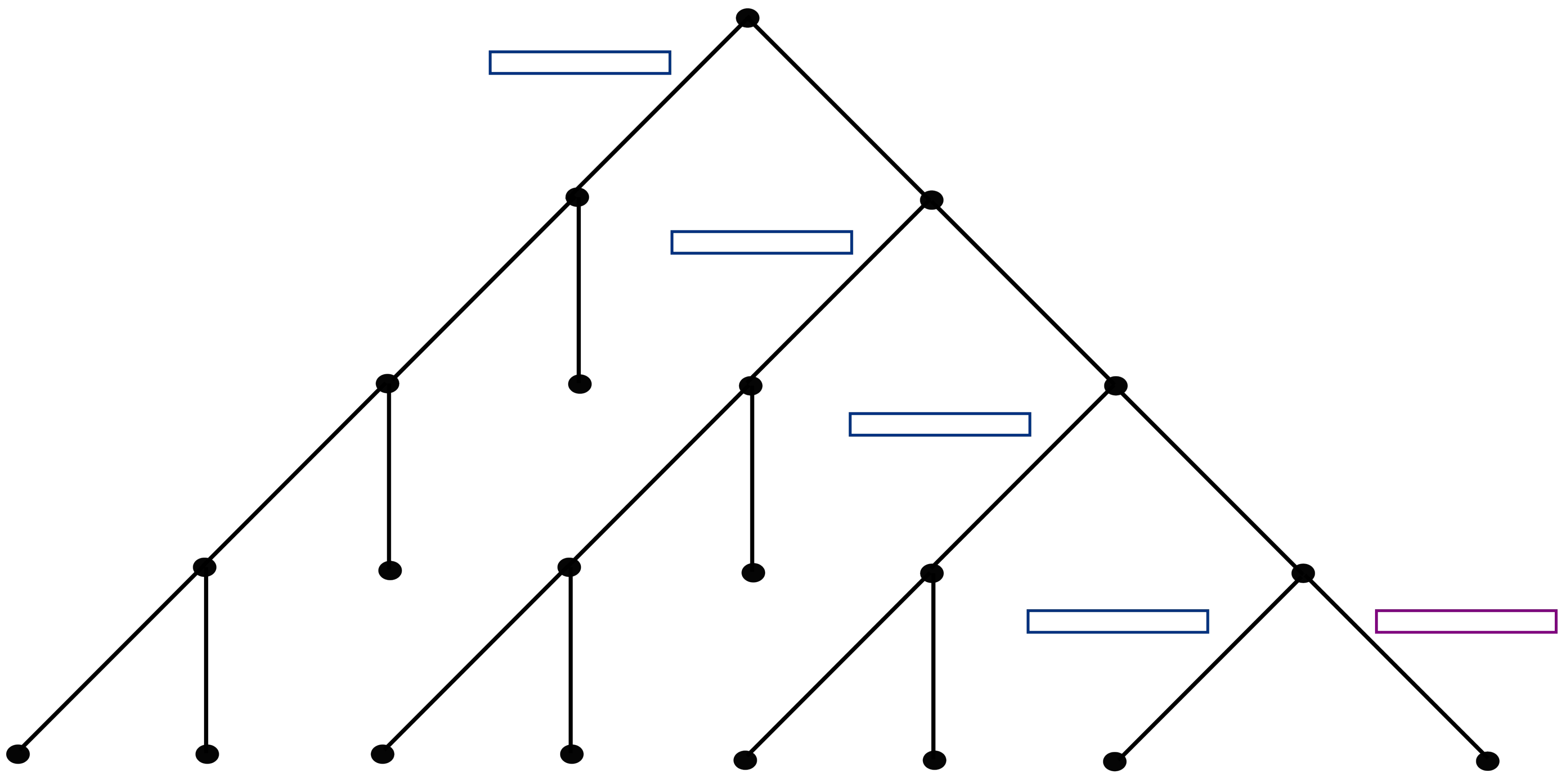


Complete Trees Do Not Give Rise to Worst Case Memory Usage

- Complete trees have the property that the worst case memory block cardinality occurs along the right-most branch.
- Worst-case memory block cardinality = 4 for both trees shown below.



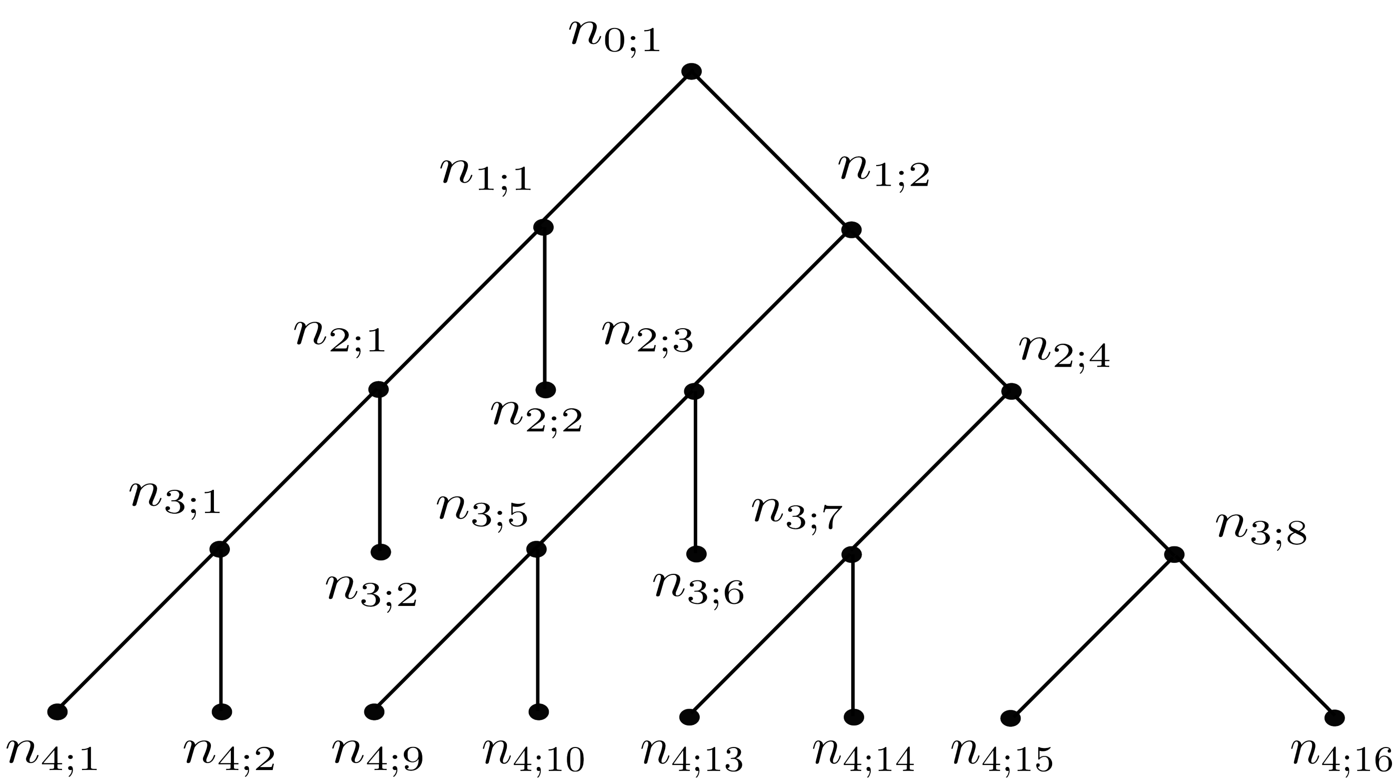
Class of 'Worst Case' Trees* Has a Surprising Structure



* K. Lessel, M. Hartman, and S. Chandrasekaran. A Fast Memory Efficient Construction Algorithm for Hierarchically Semi-Separable Representations. *Submitted to SIAM J. Matrix Analysis and Applications*

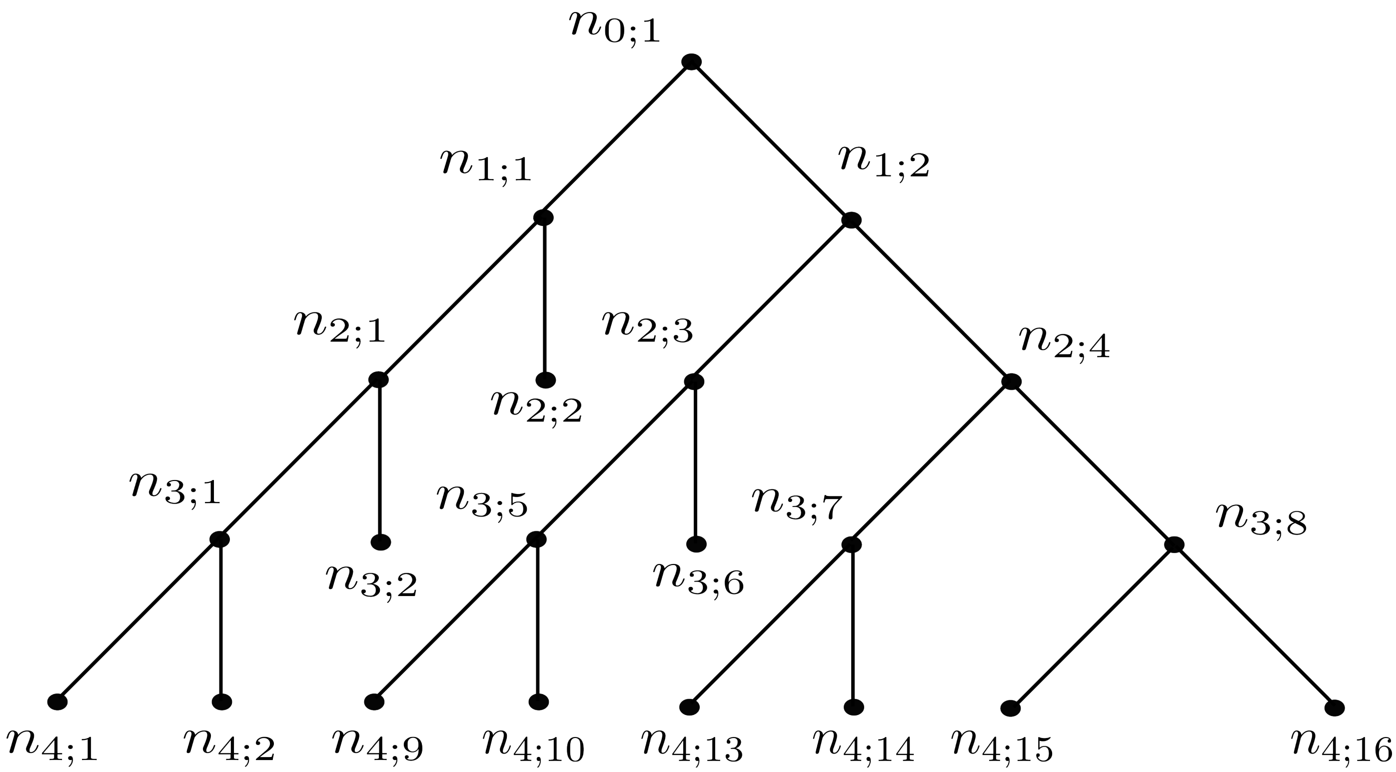
Worst Case Memory Consumption for our Algorithm

- Worst case number of memory blocks we can generate is $O(N^{0.5})$, and is generated by the binary tree with a structure as shown

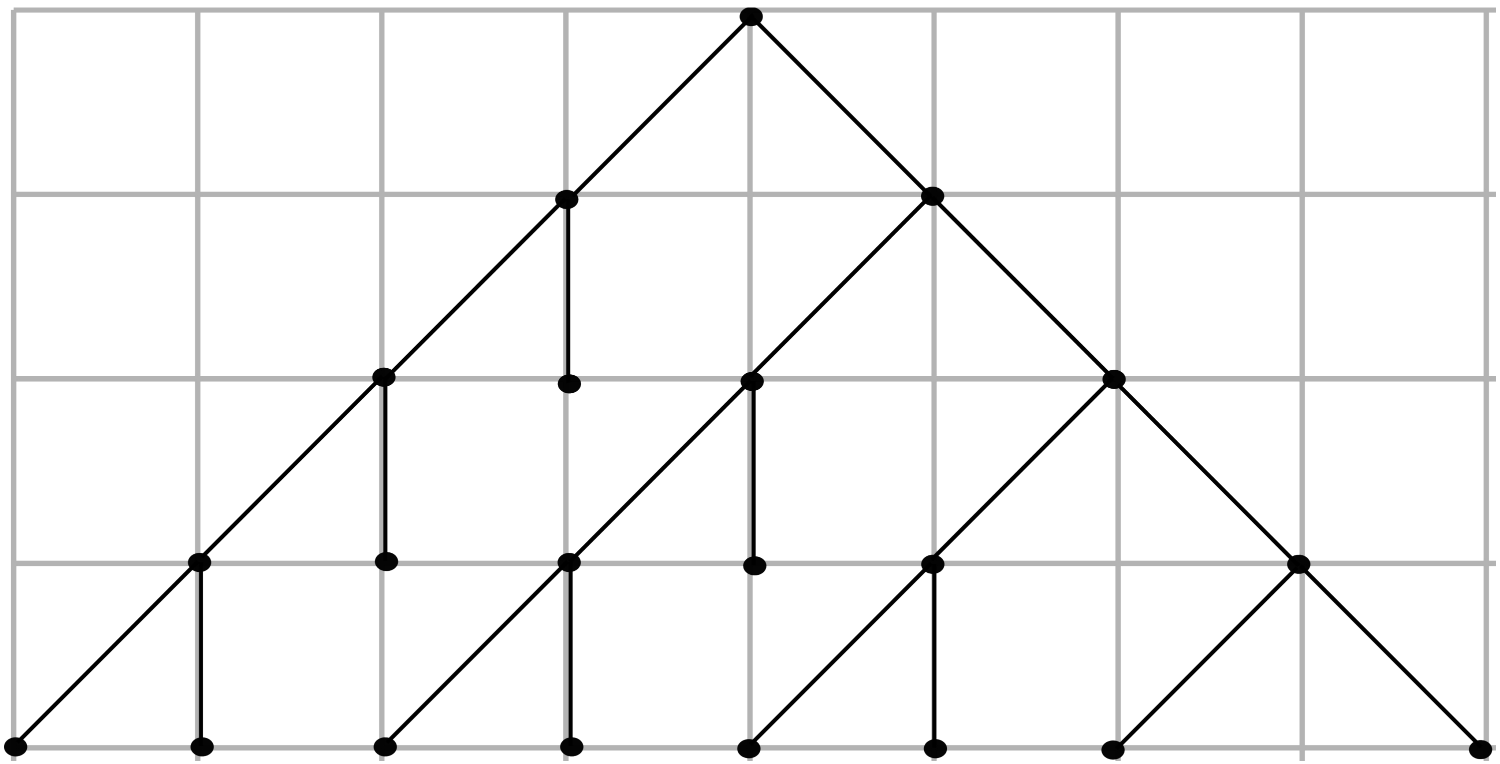


Worst Case Memory Consumption for our Algorithm

- Worst case number of memory blocks we can generate is $O(N^{0.5})$, and is generated by the binary tree with a structure as shown



Worst Case Number of Memory Blocks is $O(N^{0.5})$



Worst case number of memory blocks $= d + 1$

$$N \approx d^2$$

Worst case number of memory blocks $\approx \sqrt{N}$ *

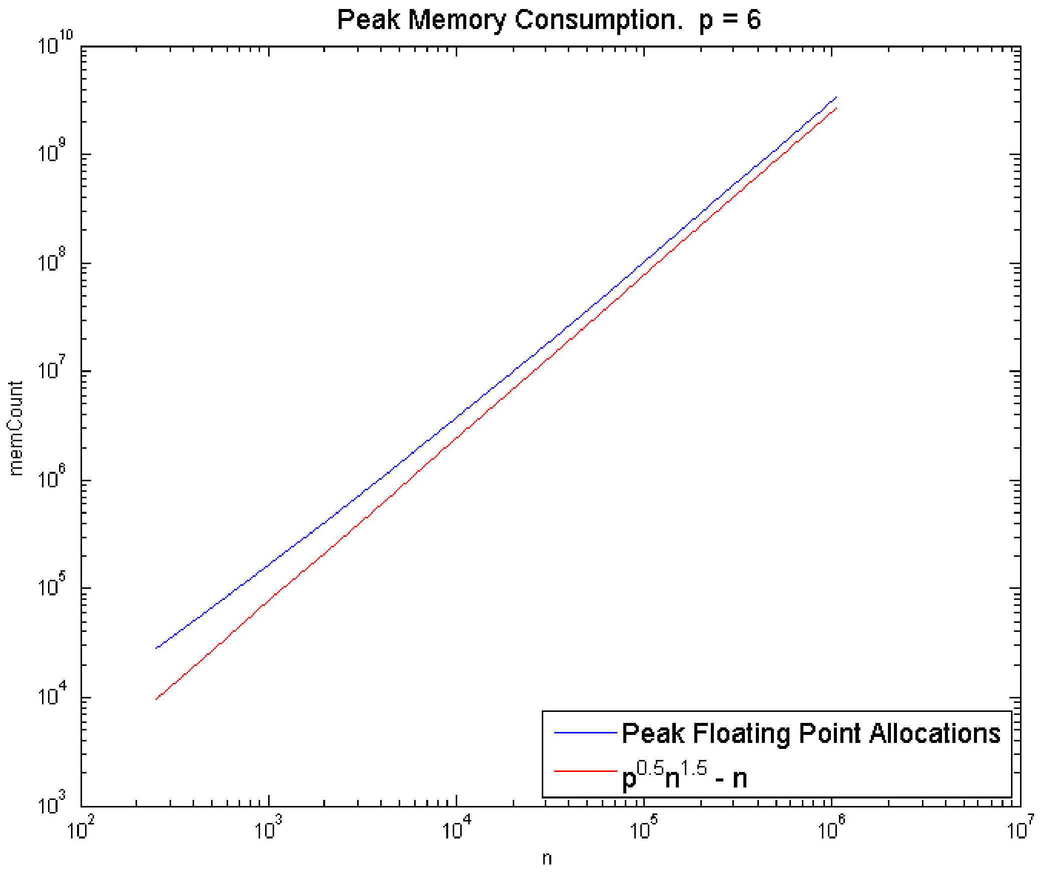
* K. Lessel, M. Hartman, and S. Chandrasekaran. A Fast Memory Efficient Construction Algorithm for Hierarchically Semi-Separable Representations. *Submitted to SIAM J. Matrix Analysis and Applications*

Relationship between the number of nodes, N and the size of our matrix, n

- Number of non-leaf nodes, N_N, is one less than the number of leaf nodes, N_L , i.e,
$$N_N = N_L - 1$$
- $N_L = n/p$
- $N = N_N + N_L$
$$= \frac{2n}{p} - 1$$
- The worst case number of memory blocks is $O(N^{0.5})$
- Peak memory consumption is $O(p^{0.5}n^{1.5})$

Numerical Results

- Upper bound for worst case peak memory consumption is $O(p^{0.5}n^{1.5})$, and we can show this is a tight bound for 'Worst Case' trees.



Conclusion

- Our 2 Phase Algorithm allows for a deepest first traversal of the HSS tree, yielding a reduction in peak memory complexity from $O(n^2)$ to $O(p^{0.5}n^{1.5})$ as compared with previous algorithms, while still taking only $O(n^2)$ flops.

- Open question: Does there exist a 'linear' memory algorithm which does not give up the $O(n^2)$ flop constraint?

References

[1] Shivkumar Chandrasekaran, Ming Gu, and Timothy Pals. A Fast ULV Decomposition Solver for Hierarchically Semiseparable Representations. *SIAM Journal on Matrix Analysis and Applications,* 28(3):603-622, 2006

[2] K. Lessel, M. Hartman, and S. Chandrasekaran. A Fast Memory Efficient Construction Algorithm for Hierarchically Semi-Separable Representations. *Submitted to SIAM J. Matrix Analysis and Applications*

[3] Per-Gunnar Martinsson. A Fast Randomized Algorithm for Computing a Hierarchically Semiseparable Representation Matrix. *SIAM Journal on Matrix Analysis and Applications,* 32(4):1251-1274, 2011.

[4] Jianlin Xia, Shivkumar Chandrasekaran, Ming Gu, and Xiaoye~S Li. Fast Algorithms for Hierarchically Semiseparable Matrices. *Numerical Linear Algebra with Applications,* 17(6):953—976, 2010.

Appendix

Why HSS?

- Matrix vector multiply: $O(n^2)$ flops vs HSS vector multiply: $O(n)$ flops
- Solution, x, of Ax = b. Gaussian Elimination: $O(n^3)$ flops vs Fast HSS solver: $O(n)$ flops

Leaf Node Computations

- For (k_1, i), (k_2, j) leaf nodes define

$$B_{k_1;i,k_2,j} = U^T_{k_1,i} A_{k_1;i,k_2,j} V_{k_2;j}$$

- For (k_1, i) a leaf node, and (k_2, j) is not, define

$$\begin{aligned} B_{k_1;i,k_2;j} &= B_{k_1;i,k_2+1;2j-1} W_{k_2+1;2j-1} \\ &\quad + B_{k_1;i,k_2+1;2j} W_{k_2+1;2j}, \end{aligned}$$

- For (k_1, i) not a leaf node, and (k_2, j) is a leaf node, define

$$\begin{aligned} B_{k_1;i,k_2;j} &= R^T_{k_1+1;2i-1} B_{k_1+1;2i-1,k_2,j} \\ &\quad + R^T_{k_1+1;2i} B_{k_1+1;2i,k_2,j}. \end{aligned}$$

Non-Leaf Node Computations

- For (k_1, i), (k_2, j) not leaf nodes, we will have $k_1 = k_2$ and can then write

$$B_{k;i,j} = B_{k_1;i,k_2;j}$$

  where $k = k_1 = k_2$

- Then define

$$\begin{aligned} B_{k;i,j} &= R^T_{k+1;2i-1} B_{k+1;2i-1,2j-1} W_{k+1;2j-1} \\ &+ R^T_{k+1;2i-1} B_{k+1;2i-1,2j} W_{k+1;2j} \\ &+ R^T_{k+1;2i} B_{k+1;2i,2j-1} W_{k+1;2j-1} \\ &+ R^T_{k+1;2i} B_{k+1;2i,2j} W_{k+1;2j}\,, \end{aligned}$$

Phase 1 - Leaf Node Computations

$$
{}_r\acute{H}_{k;i} = \begin{pmatrix} U_{k;i} & * \end{pmatrix} \begin{pmatrix} \Sigma_{k;i} & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} Q_{k;i}^T \\ * \end{pmatrix}
$$

$$
{}_c\acute{H}_{k;i} = \begin{pmatrix} P_{k;i} & * \end{pmatrix} \begin{pmatrix} \Lambda_{k;i} & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} V_{k;i}^T \\ * \end{pmatrix}.
$$

Phase 1 – Non-leaf Node Computations

$$ {}_r\tilde{H}_{k;2i-1} = \Sigma_{k;2i-1}\, Q_{k;2i-1}^T \qquad {}_c\tilde{H}_{k;2i-1} = P_{k;2i-1}\, \Lambda_{k;2i-1} $$

$$ {}_r\tilde{H}_{k;2i} = \Sigma_{k;2i}\, Q_{k;2i}^T \qquad {}_c\tilde{H}_{k;2i} = P_{k;2i}\, \Lambda_{k;2i} $$

Phase 1 – Non-leaf Node Computations

- Remove block cloumns of $Q_{k;i,j}^T$ which corresond to the columns that lie in the diagonal block $D_{k-1;i}$

$$\mathrm{Q}_{k;i}^T = \left(\begin{array}{cccc} Q_{k;i,1}^T & Q_{k;i,2}^T & \cdots & Q_{k;i,2^k-1}^T \end{array} \right)$$

$$\tilde{Q}_{k;i}^T = \left(\begin{array}{cccccc} Q_{k;i,1}^T & \cdots & Q_{k;i,i-1}^T & Q_{k;i,i+1}^T & \cdots & Q_{k;i,2^k-1}^T \end{array} \right)$$

- Compressed Hankel blocks at node $(k-1;i)$

$${}_rH_{k-1;i} = \left(\begin{array}{c} \Sigma_{k;2i-1}\,\tilde{Q}_{k;2i-1}^T \\ \Sigma_{k,2i}\,\tilde{Q}_{k;2i}^T \end{array} \right)$$

$${}_cH_{k-1;i} = \left(\begin{array}{cc} \tilde{P}_{k;2i-1}\,\Lambda_{k;2i-1} & \tilde{P}_{k;2i}\,\Lambda_{k;2i} \end{array} \right)$$

Algorithm 1 Pass 1U

Algorithm 1 Pass 1U - Memory Efficient HSS Algorithm

function HSS_BASIS($tree$)
    if $tree$ is a leaf node **then**
        $(U_{k;i}, \Sigma_{k;i}, Q^T_{k;i}) = \text{TRUNC\_SVD}([A_{k;i,1} \;\; A_{k;i,2} \;\; \dots \;\; A_{k;i,i-1} \;\; A_{k;i,i+1} \;\; \dots \;\; A_{k;i,end}])$
        return $\Sigma_{k;i}\tilde{Q}^T_{k;i}$ ▷ $\tilde{Q}_{k;i}$ is defined as previously stated in equation (3.8)
    else ▷ $tree$ is **not** a leaf node
        $\_,\ treeL,\ treeR = tree$
        if DEPTH($treeL$) $\geq$ DEPTH($treeR$) **then**
            ${}_rH_{k+1;2i-1}$ = HSS_BASIS($treeL$)
            ${}_rH_{k+1;2i}$ = HSS_BASIS($treeR$)
        else
            ${}_rH_{k+1;2i}$ = HSS_BASIS($treeR$)
            ${}_rH_{k+1;2i-1}$ = HSS_BASIS($treeL$)
        end if
        ${}_rH_{k;i} = \begin{pmatrix} {}_rH_{k+1;2i-1} \\ {}_rH_{k+1;2i} \end{pmatrix}$
        ${}_rH_{k+1;2i-1} = ();\quad {}_rH_{k+1;2i} = ()$
        $\begin{pmatrix} R_{k+1;2i-1} \\ R_{k+1;2i} \end{pmatrix},\ \Sigma_{k;i},\ X_{k;i} = \text{TRUNC\_SVD}({}_rH_{k;i})$
        return $\Sigma_{k;i}\tilde{X}^T_{k;i}$ ▷ $\tilde{X}_{k;i}$ is defined as previously stated in equation (3.8)
    end if
end function

Algorithm 2 Pass 2BU

Algorithm 2 Pass 2BU - Computation of Expansion Coefficients ($B_{k;i-1,i}$) Corresponding to Diagonal Blocks

Require: *tree* is not a leaf node

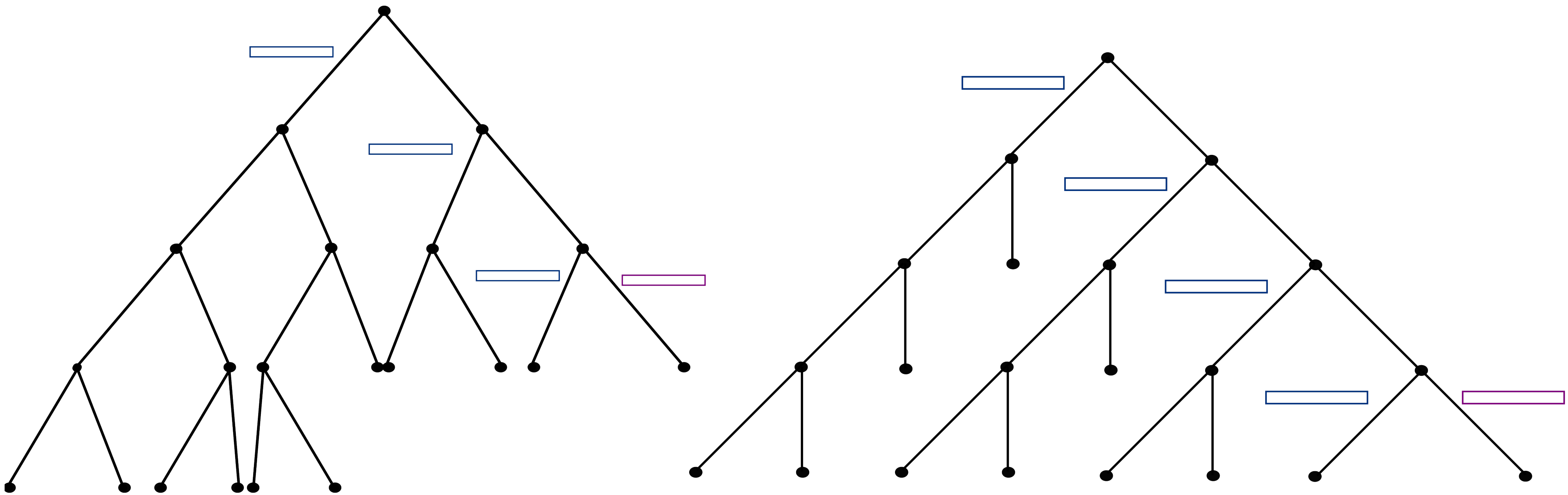
function B_DIAG(tree)
  $-,\ treeL,\ treeR = tree$ ▷ $\exists\ (k1; i)$ s.t. it is the numbering for the root node of $treeL$.
  ▷ $\exists\ (k2; j)$ s.t. it is the numbering for the root node of $treeR$.
  if $treeL$ is a leaf node and $treeR$ is a leaf node **then**
    $B_{k_1;i,k_2;j} = U^T_{k_1;i} A_{k_1;i,k_2;j} V_{k_2;j}$
  else if $treeL$ is **not** a leaf node and $treeR$ is **not** a leaf node **then**
    $B_{k_1;i,k_2;j} =$ B_OFFDIAG($treeL$, $treeR$)
    B_DIAG($treeL$)
    B_DIAG($treeR$)
  else if $treeL$ is a leaf node and $treeR$ is **not** a leaf node **then**
    $-,\ treeRL\ \ treeRR = treeR$
    $B_{k_1;i,k_2+1;2j-1} =$ B_OFFDIAG($treeL$, $treeRL$)
    $B_{k_1;i,k_2+1;2j} =$ B_OFFDIAG($treeL$, $treeRR$)
    $B_{k_1;i,k_2;j} = B_{k_1;i,k_2+1;2j-1} W_{k_2+1;2j-1} + B_{k_1;i,k_2+1;2j} W_{k_2+1;2j}$
    $B_{k_1;i,k_2+1;2j-1} = ();\qquad B_{k_1;i,k_2+1;2j} = ()$
    B_DIAG($treeR$)
  else if $treeL$ is **not** a leaf node and $treeR$ is a leaf node **then**
    $-,\ treeLL,\ treeLR = treeL$
    $B_{k_1+1;2i-1,k_2,j} =$ B_OFFDIAG($treeLL$, $treeR$)
    $B_{k_1+1;2i,k_2,j} =$ B_OFFDIAG($treeLR$, $treeR$)
    $B_{k_1;i,k_2;j} = R^T_{k_1+1;2i-1} B_{k_1+1;2i-1,k_2;j} + R^T_{k_1+1;2i} B_{k_1+1;2i,k_2;j}$
    $B_{k_1+1;2i-1,k_2;j} = ();\qquad B_{k_1+1;2i,k_2;j} = ()$
    B_DIAG($treeL$)
  end if
end function

Algorithm 3 Pass 2BU

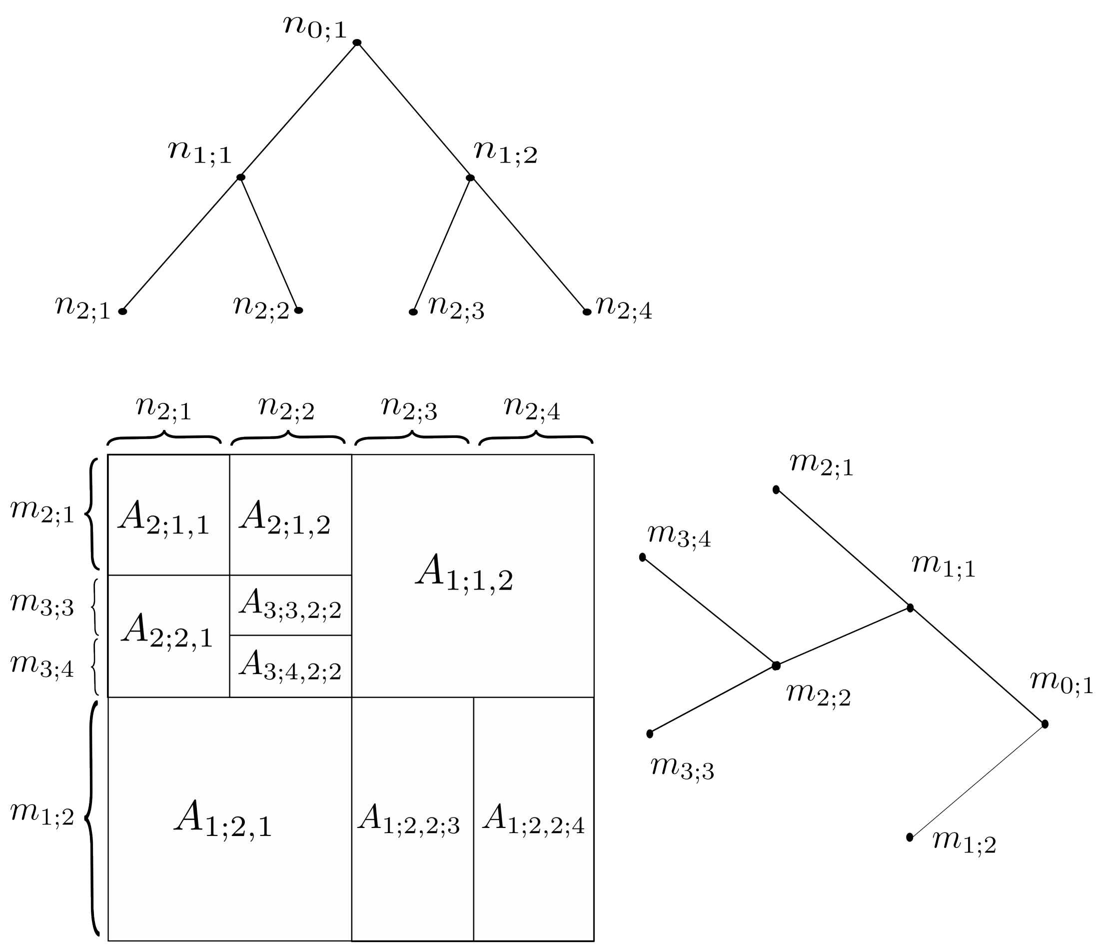
Algorithm 3 Pass 2BU - Computation of Expansion Coefficients ($B_{k;i-1,i}$) Corresponding to Off-Diagonal Blocks

function B_OFFDIAG(treeL,treeR)
  $-,\ treeLL,\ treeLR = treeL$
  $-,\ treeRL,\ treeRR = treeR$
  if $treeL$ is a leaf node and $treeR$ is a leaf node **then**
    $B_{k_1;i,k_2;j} = U^T_{k_1;i} A_{k_1;i,k_2;j} V_{k_2;j}$
    return $B_{k_1;i,k_2;j}$
  else if $treeL$ is **not** a leaf node and $treeR$ is **not** a leaf node **then**
    $B_{k_1+1;2i-1,k_2+1;2j-1}$ = B_OFFDIAG($treeLL$,$treeRL$)
    $B_{k_1+1;2i-1,k_2+1;2j-}$ = B_OFFDIAG($treeLL$,$treeRR$)
    $B_{k_1+1;2i,k_2+1;2j-1}$ = B_OFFDIAG($treeLR$,$treeRL$)
    $B_{k_1+1;2i,k_2+1;2j}$ = B_OFFDIAG($treeLR$,$treeRR$)
    $$B_{k_1;i,k_2;j} = R^T_{k_1+1;2i-1} B_{k_1+1;2i-1,k_2+1;2j-1} W_{k_2+1;2j-1} + R^T_{k_1+1;2i-1} B_{k_1+1;2i-1,k_2+1;2j} W_{k_2+1;2j} + R^T_{k_1+1;2i} B_{k_1+1;2i,k_2+1;2j-1} W_{k_2+1;2j-1} + R^T_{k_1+1;2i} B_{k_1+1;2i,k_2+1;2j} W_{k_2+1;2j}$$
    $B_{k_1+1;2i-1,k_2+1;2j-1} = ()$; $B_{k_1+1;2i-1,k_2+1;2j} = ()$;
    $B_{k_1+1;2i,k_2+1;2j-1} = ()$; $B_{k_1+1;2i,k_2+1;2j} = ()$
    return $B_{k_1;i,k_2;j}$
  else if $treeL$ is a leaf node and $treeR$ is **not** a leaf node **then**
    $B_{k_1;i,k_2+1;2j-1}$ = B_OFFDIAG($treeL$,$treeRL$)
    $B_{k_1;i,k_2+1;2j}$ = B_OFFDIAG($treeL$,$treeRR$)
    $B_{k_1;i,k_2;j} = B_{k_1;i,k_2+1;2j-1} W_{k_2+1;2j-1} + B_{k_1;i,k_2+1;2j} W_{k_2+1;2j}$
    $B_{k_1;i,k_2+1;2j-1} = ()$; $B_{k_1;i,k_2+1;2j} = ()$
    return $B_{k_1;i,k_2;j}$
  else if $treeL$ is **not** a leaf node and $treeR$ is a leaf node **then**
    $B_{k_1+1;2i-1,k_2;j}$ = B_OFFDIAG($treeLL$,$treeR$)
    $B_{k_1+1;2i,k_2;j}$ = B_OFFDIAG($treeLR$,$treeR$)
    $B_{k_1;i,k_2;j} = R^T_{k_1+1;2i-1} B_{k_1+1;2i-1,k_2;j} + R^T_{k_1+1;2i} B_{k_1+1;2i,k_2;j}$
    $B_{k_1+1;2i-1,k_2;j} = ()$; $B_{k_1+1;2i,k_2;j} = ()$
    return $B_{k_1;i,k_2;j}$
  end if
end function

Worst Case Memory Consumption for our Algorithm



Example Block Partitioning of a Matrix with Corresponding Partition Trees



Future Work

- Fast Multipole Method (FMM) construction Algorithm
- FMM x FMM
- Application to classical HSS algorithms: HSS Multiply & HSS Solver