Super-fast Solvers for FMM Matrices

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Fast Multi-pole Method

- Greengard & Rokhlin (1985)
- Fast matrix multiplication algorithm
- Exploits low numerical-rank in sub-matrices

Ubiquitous

- Discrete integral equations
- Schur complements of discrete differential equations
- Many sparse matrices
- Cauchy-like matrices

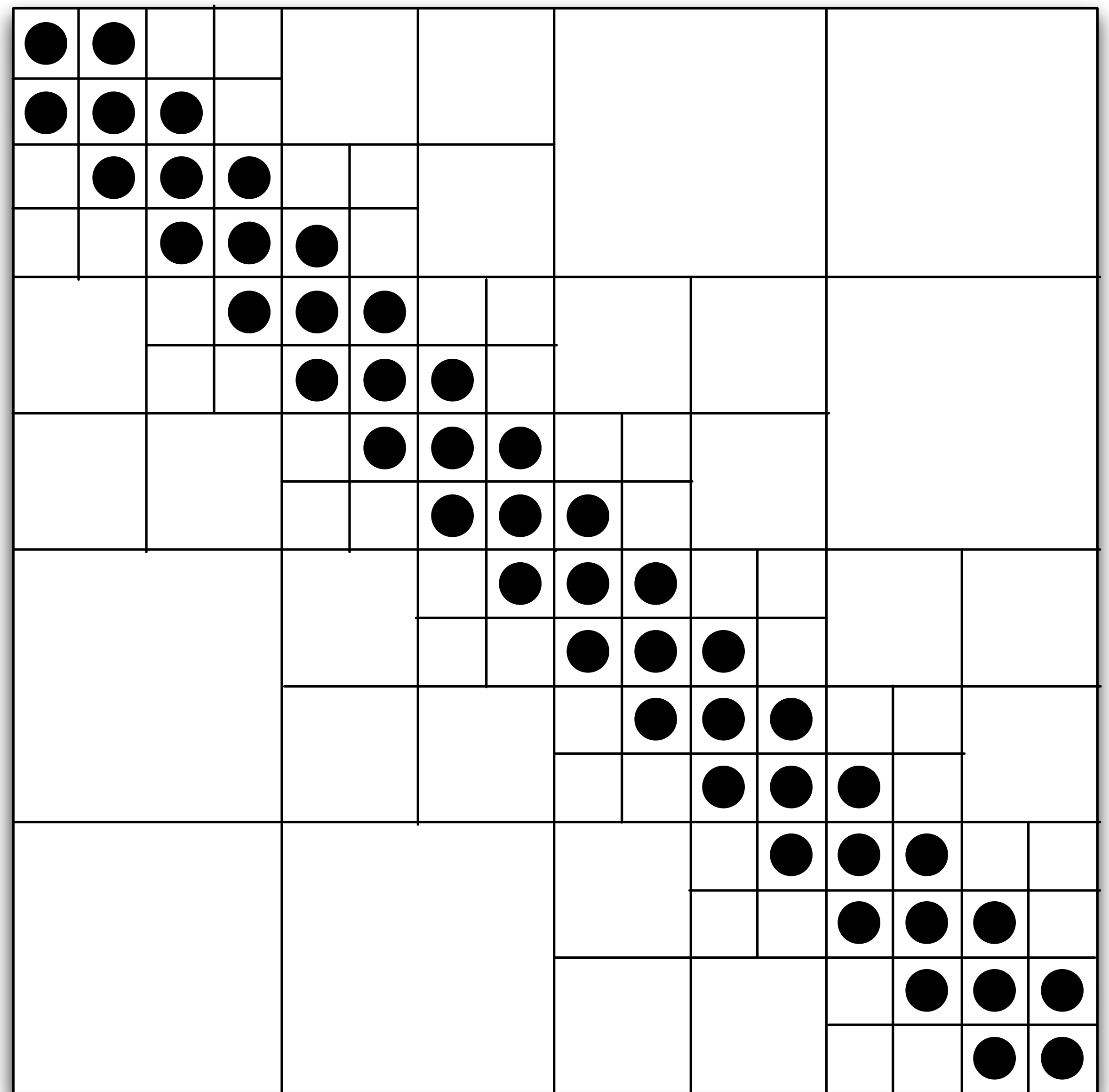
Picture Gallery of FMM Matrices

1 spatial dimension

Blank squares have
constant numerical rank

$A = \log ||x_i - x_j||$

$x_i \in \mathcal{R}$

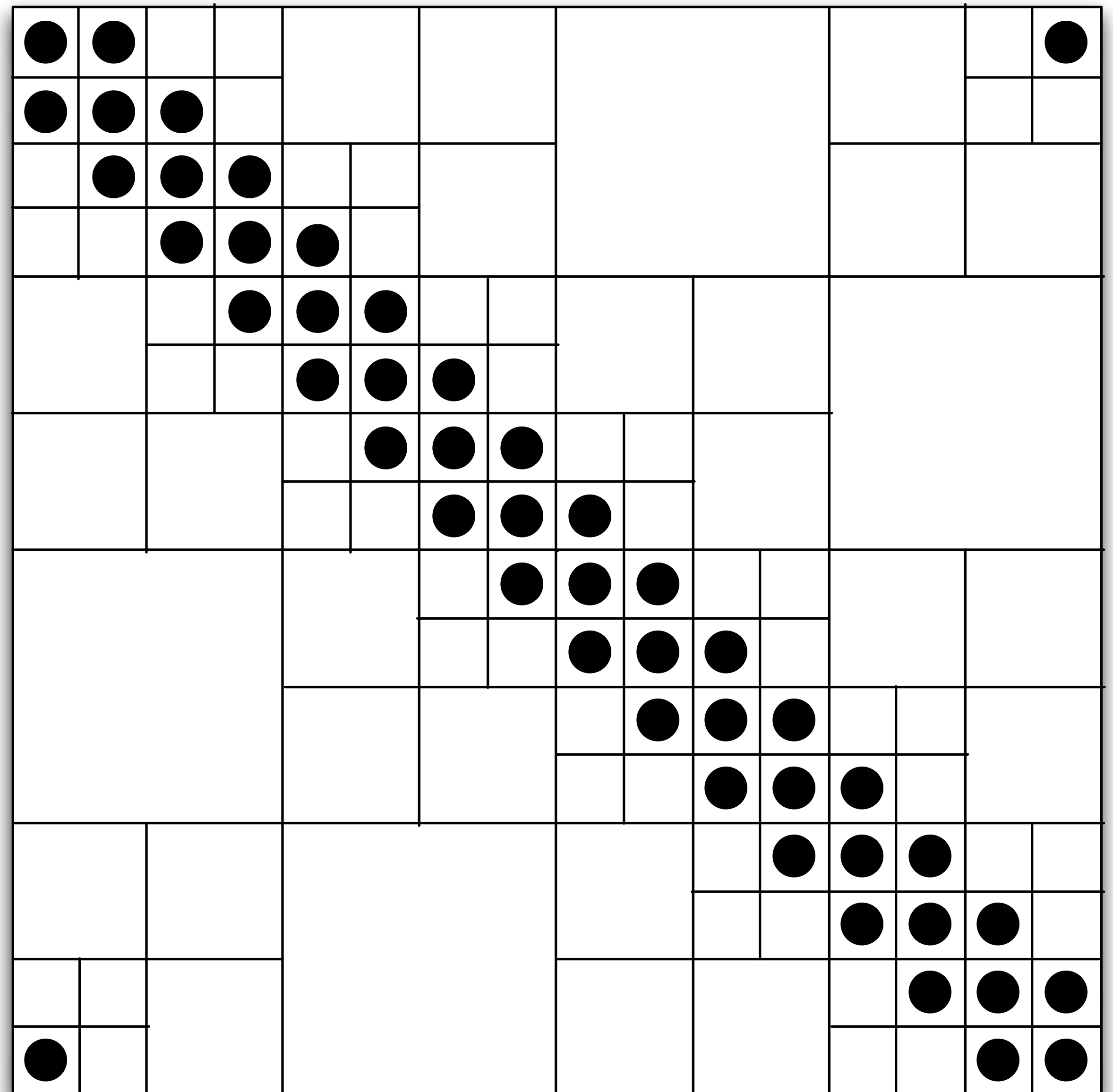


1½ spatial dimensions

Blank squares have constant numerical rank

$$A = \log ||z_i - z_j||$$

$$z_j = e^{i\theta_j}$$

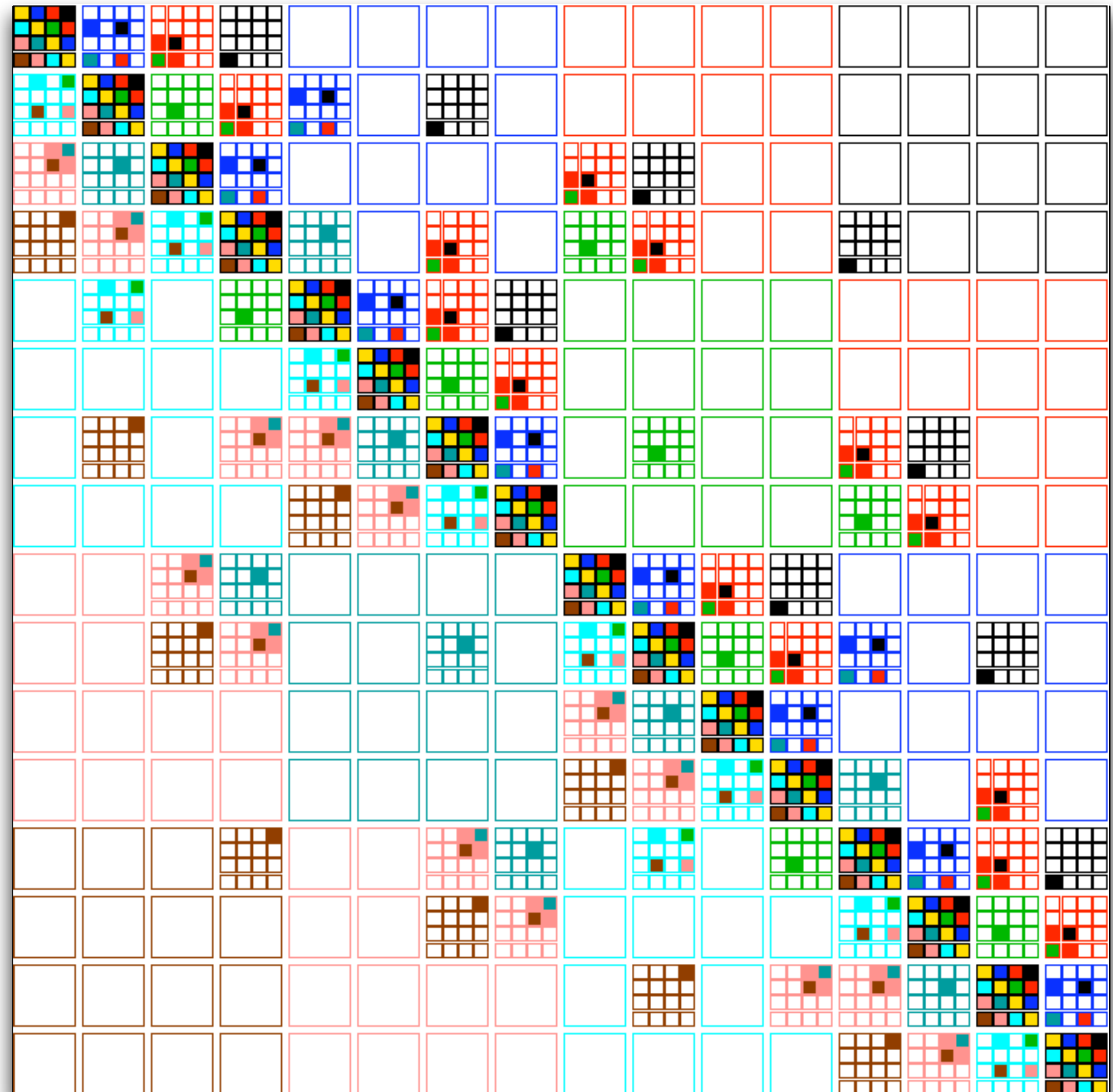


2 spatial dimensions

Blank squares have
constant numerical rank

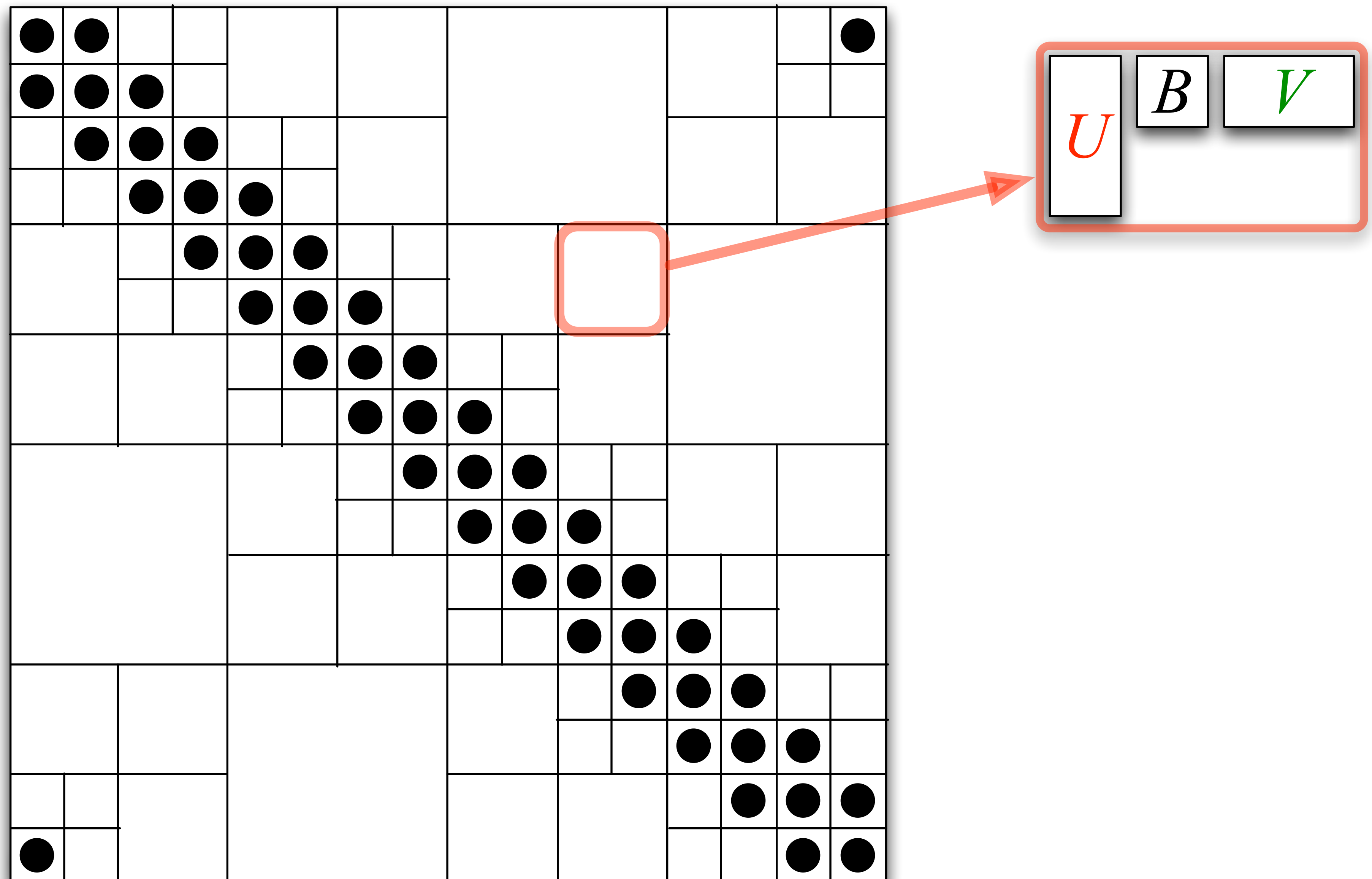
$$A = \log ||z_i - z_j||$$

$$z_j \in \mathcal{R}^2$$



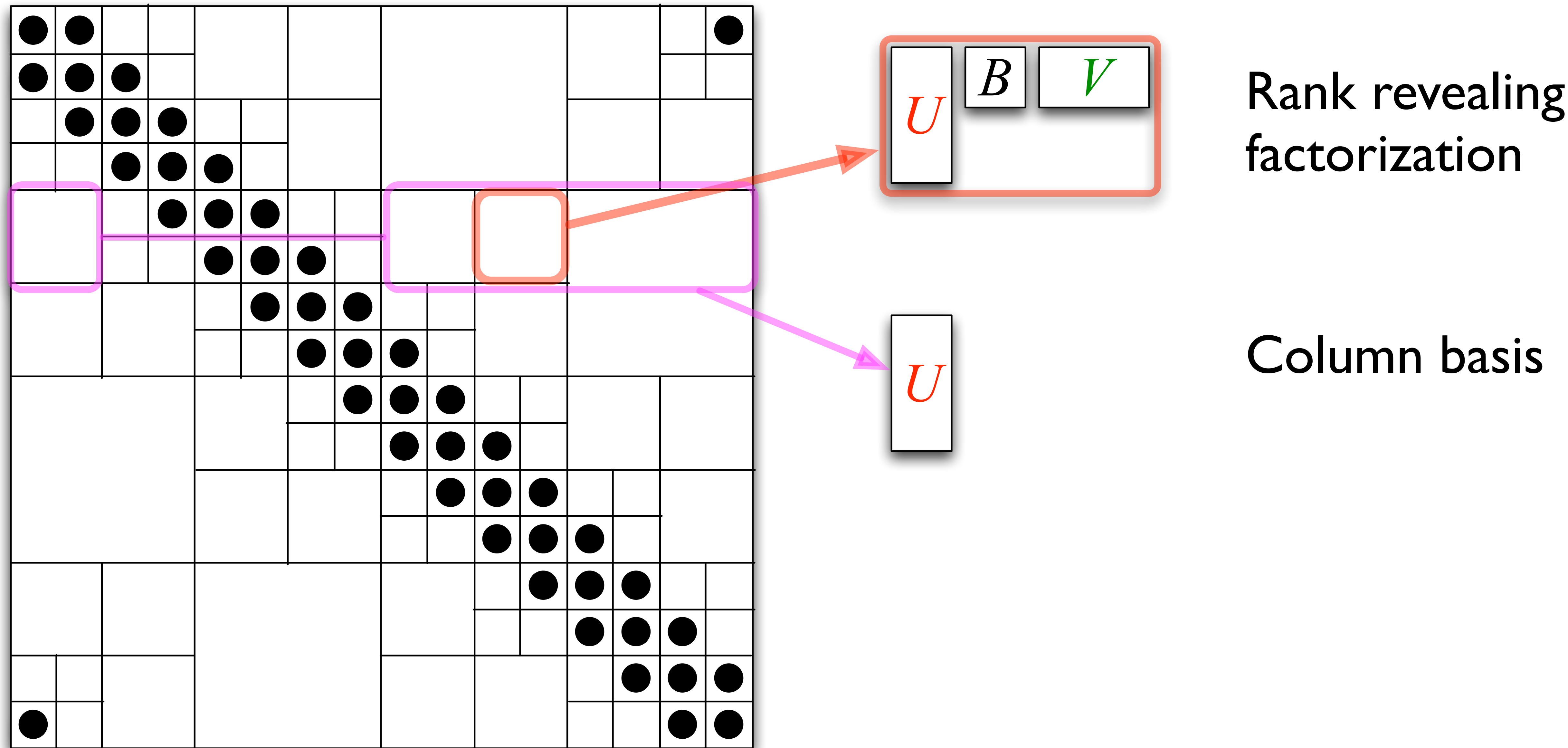
Algebraic
FMM Representation

FMM Representation

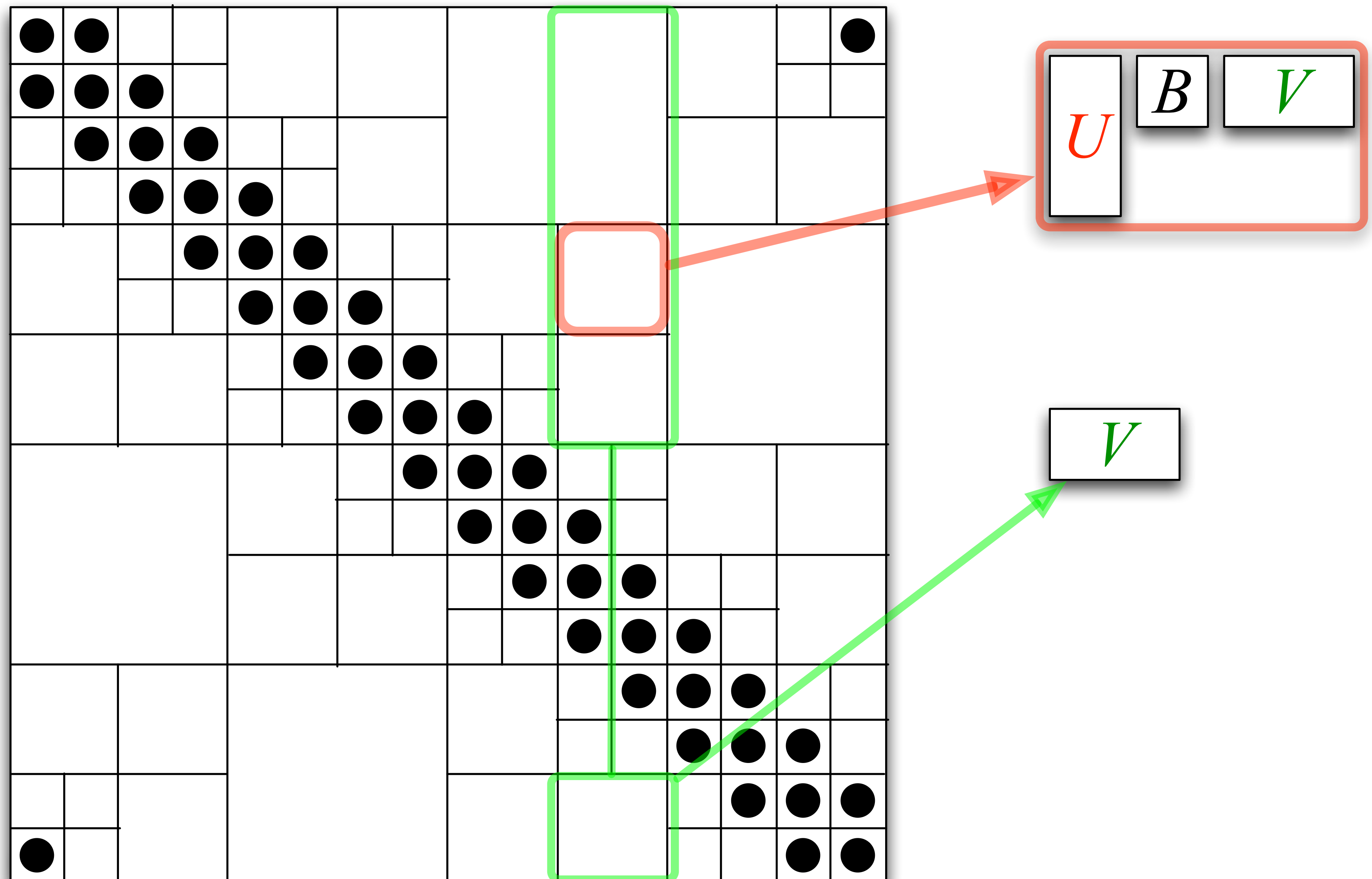


Rank revealing factorization

FMM Representation



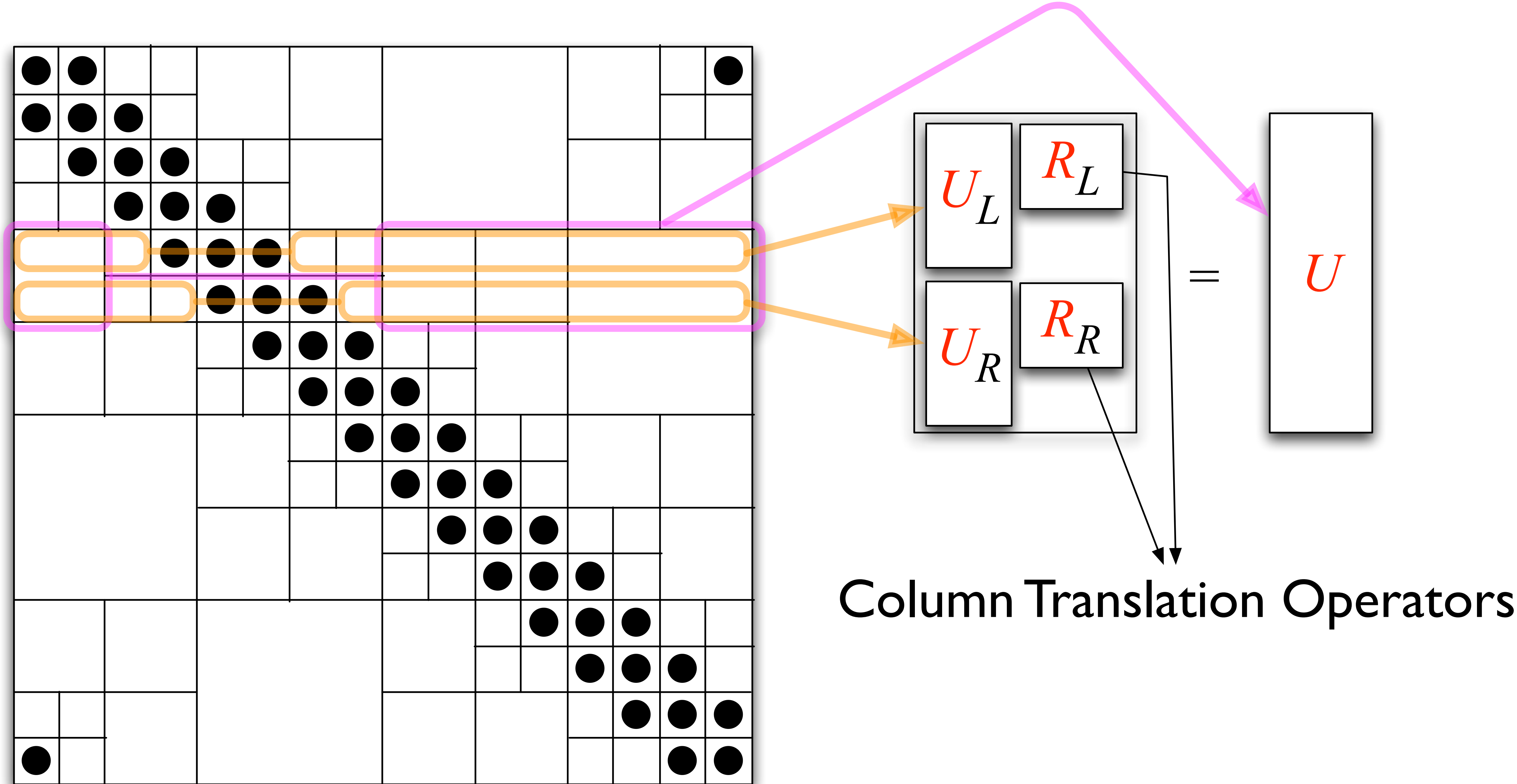
FMM Representation



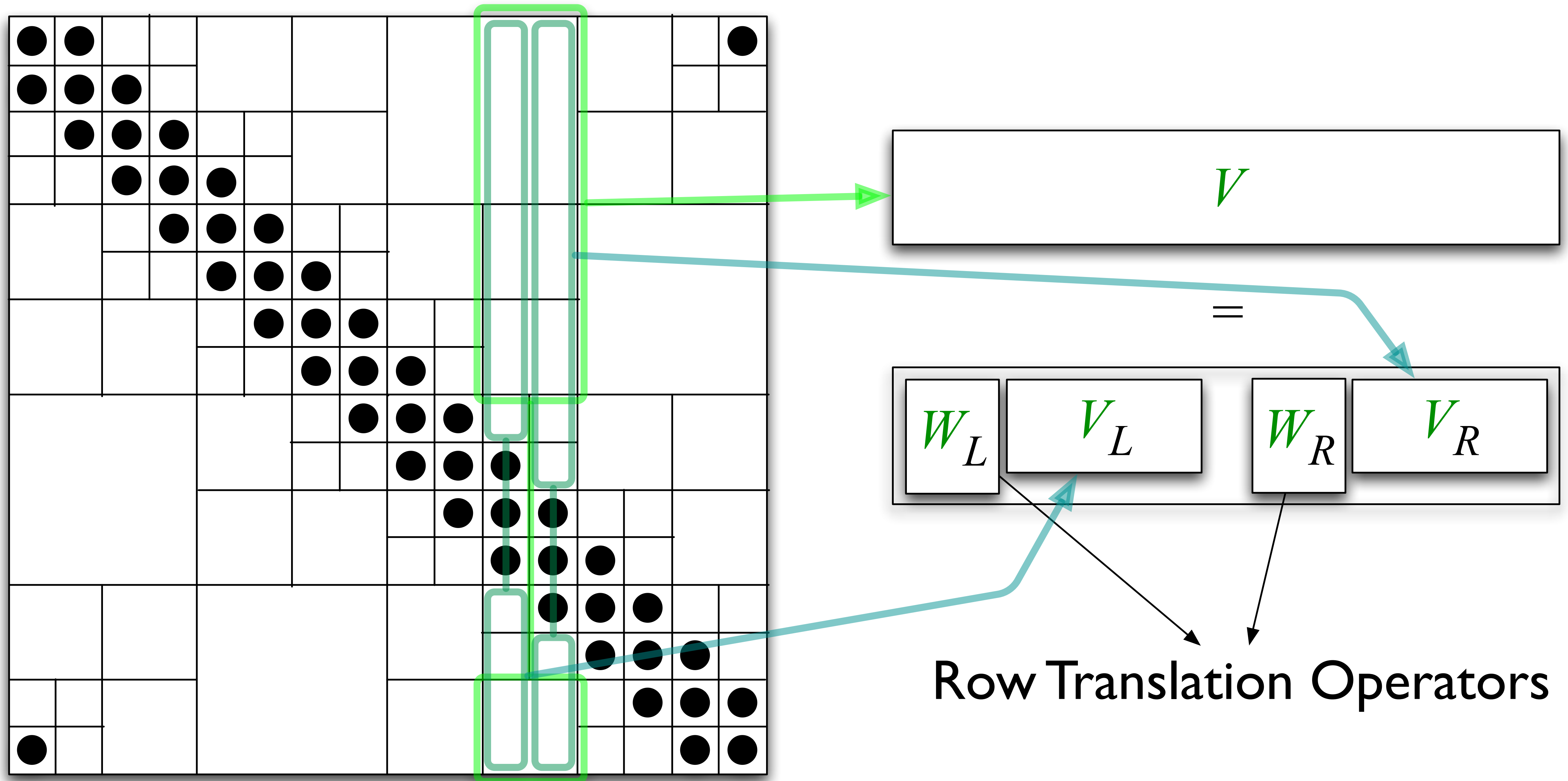
Rank revealing factorization

Row basis

FMM Representation



FMM Representation



Fast algorithms : history

- Linear-time construction for important kernels, Rokhlin *et al.*
- Fast multiplication algorithm, Greengard & Rokhlin
- Fast direct solvers, Rokhlin *et al.*

Fast algorithms : research programme

- Super-fast LU solvers
- Super-fast orthogonal solvers
- Super-fast direct sparse solvers
- Optimal pre-conditioners

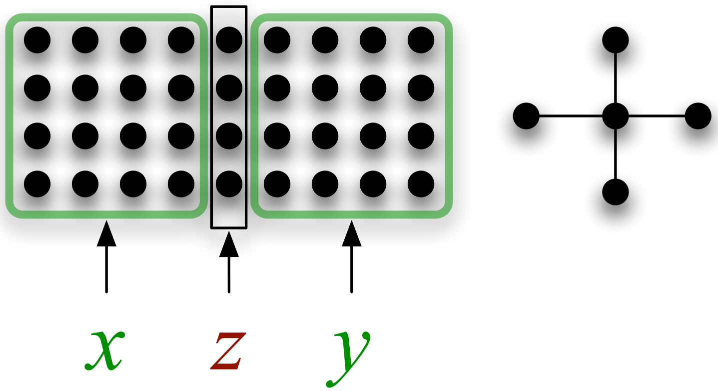
Numerical Experiments

Our current prototypes are twice as fast as the standard direct sparse solvers on two dimensional elliptic PDEs on 1024 x 1024 grids.

$$\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial}{\partial x}u(x,y)\right)+\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial}{\partial y}u(x,y)\right)=f(x,y)$$

p,q ➠	1	piece-wise constant	$10^{-7}+\sin(\varrho(x^2+y^2))$	random

Fill-in is FMM



$$\begin{bmatrix} A_{xx} & & A_{xz} \\ & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Schur complement

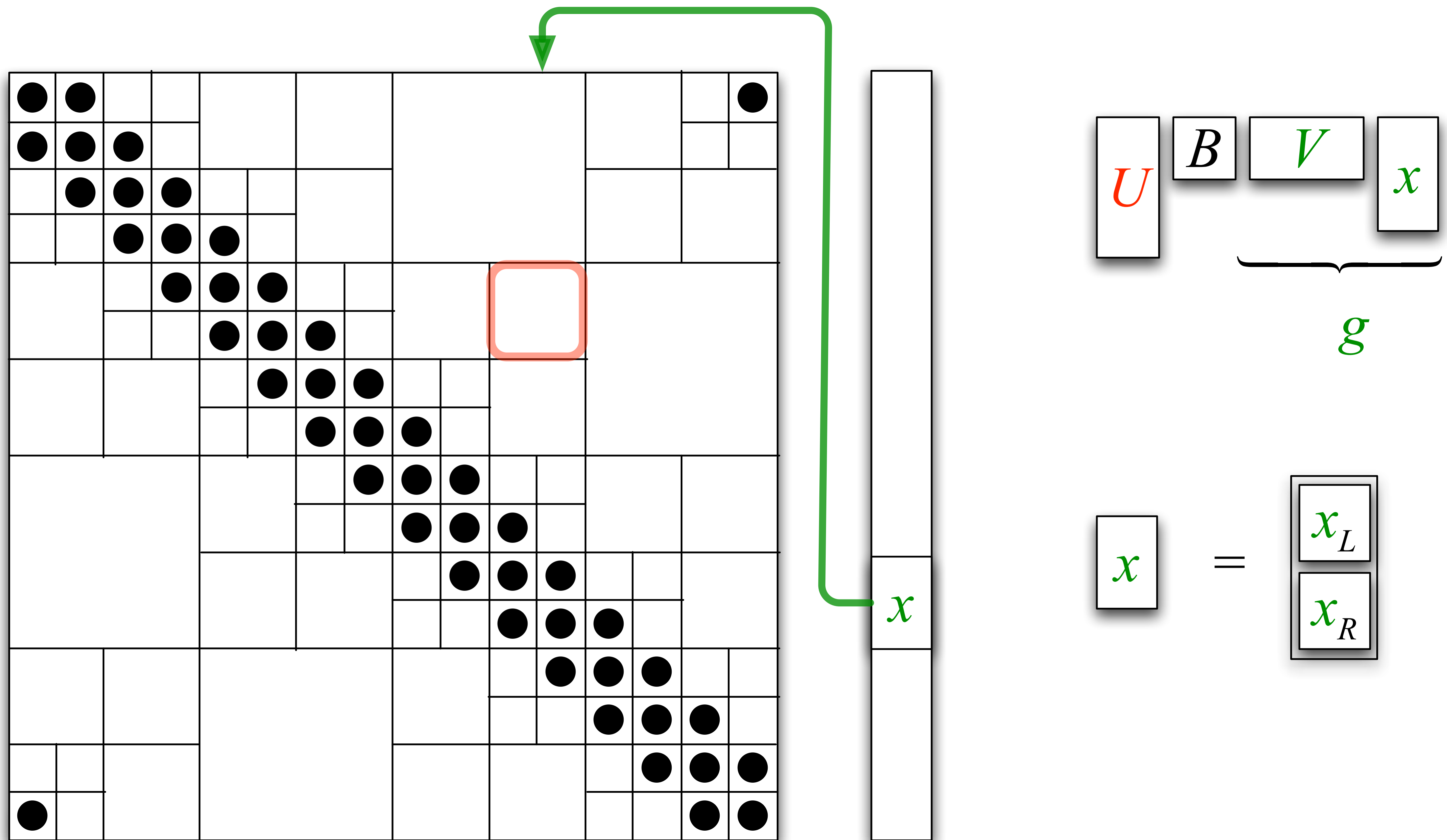
$$S = A_{zz} - A_{zx} A_{xx}^{-1} A_{xz} - A_{zy} A_{yy}^{-1} A_{yz}$$

S^{-1} can be viewed as the restriction of the Green's function to the interface

Hence it must have a compact FMM representation

Fast Multiplication

FMM



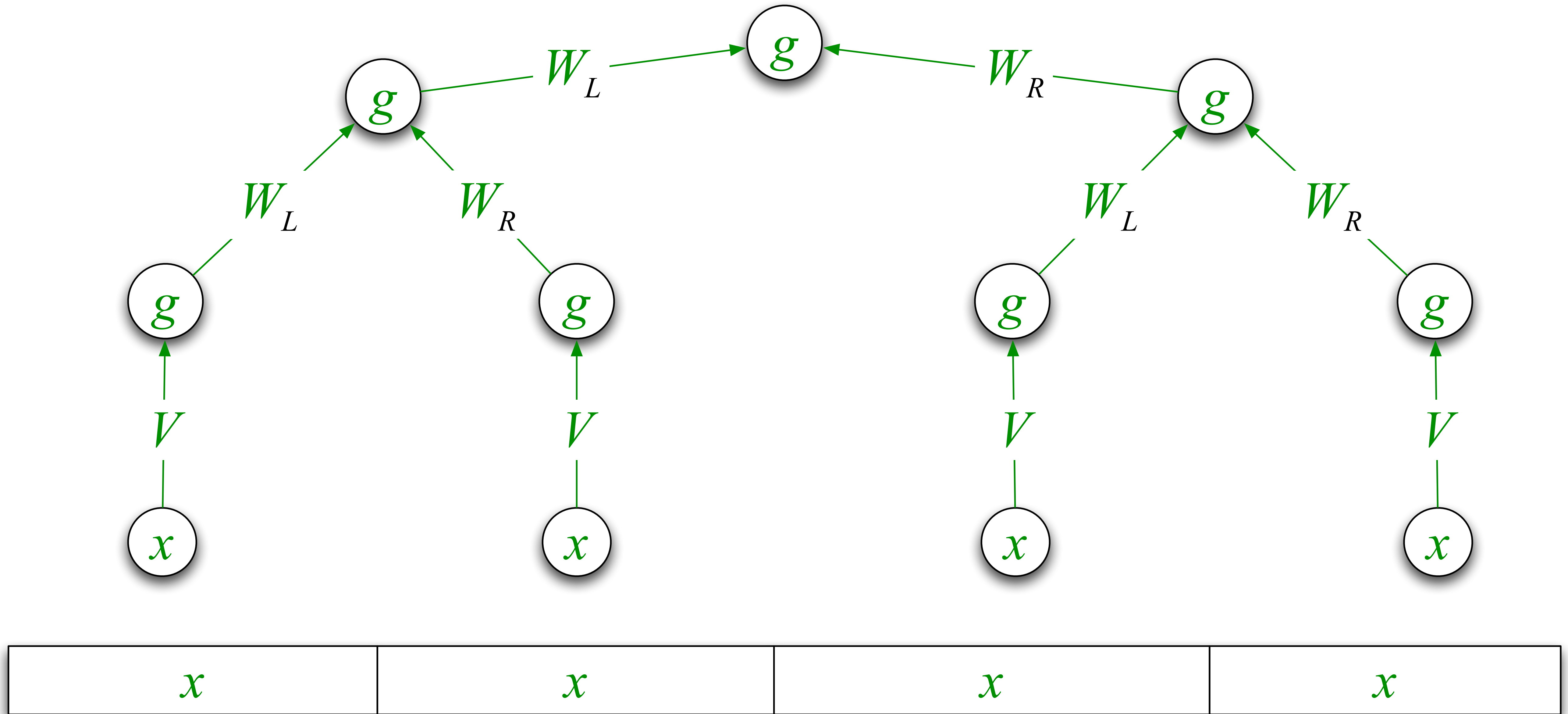
FMM Up Sweep Recursions

$$V \, x = g$$

$$= \begin{bmatrix} W_L V_L & W_R V_R \end{bmatrix} \begin{bmatrix} x_L \\ x_R \end{bmatrix}$$

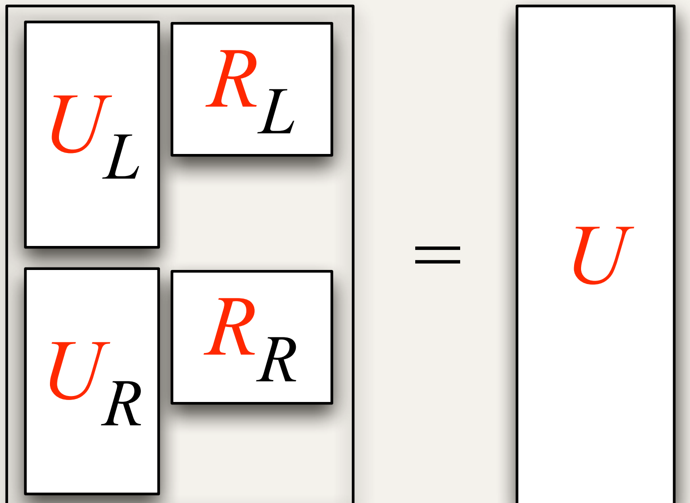
$$= W_L \, g_L + W_R \, g_R = g$$

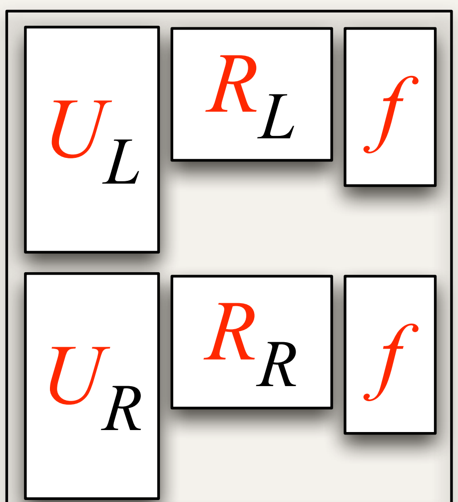
FMM Up Sweep Recursions

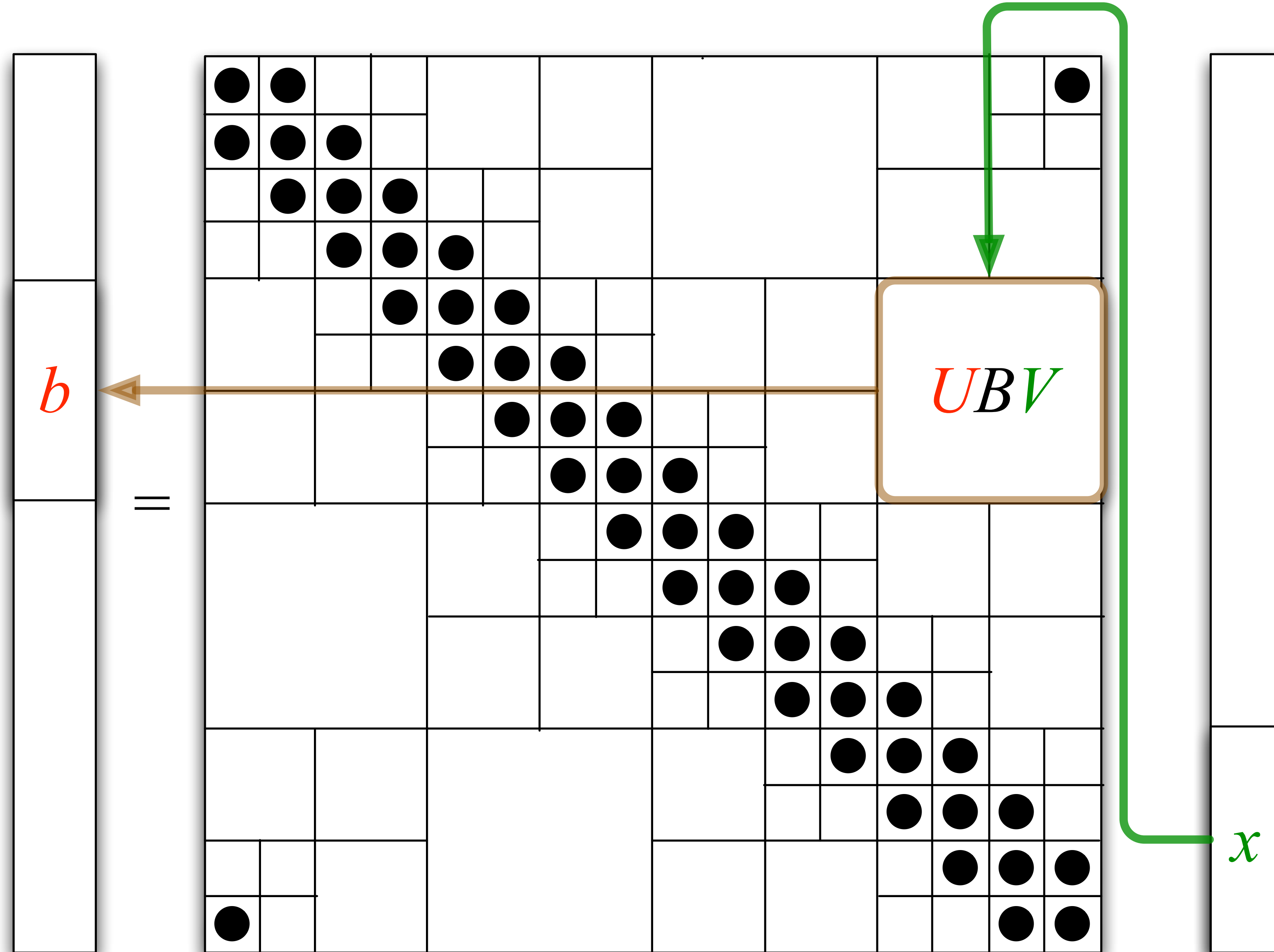


FMM

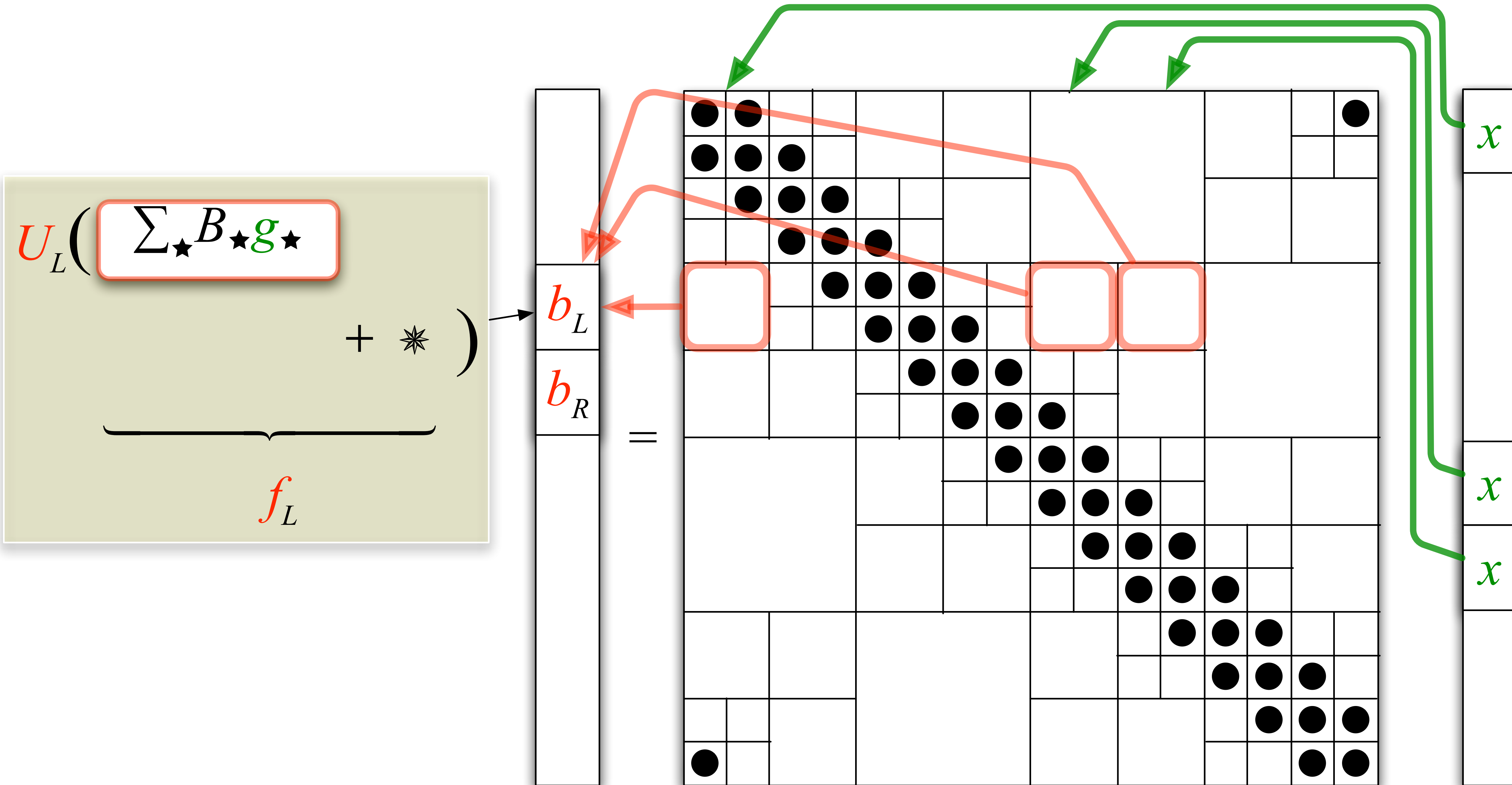
$$U\,B\,V\,x = U\,\underbrace{B\,g}_{f}$$



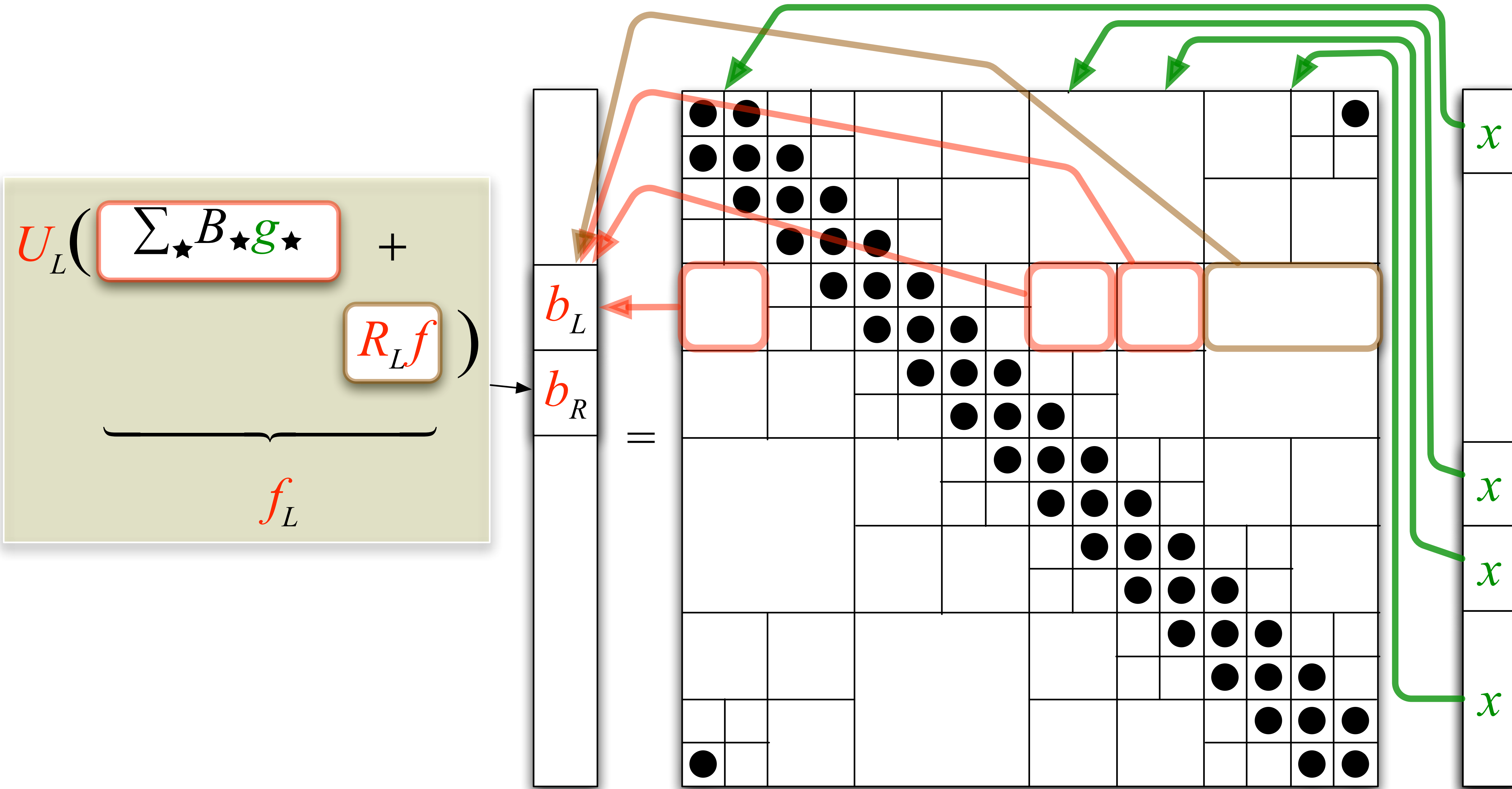




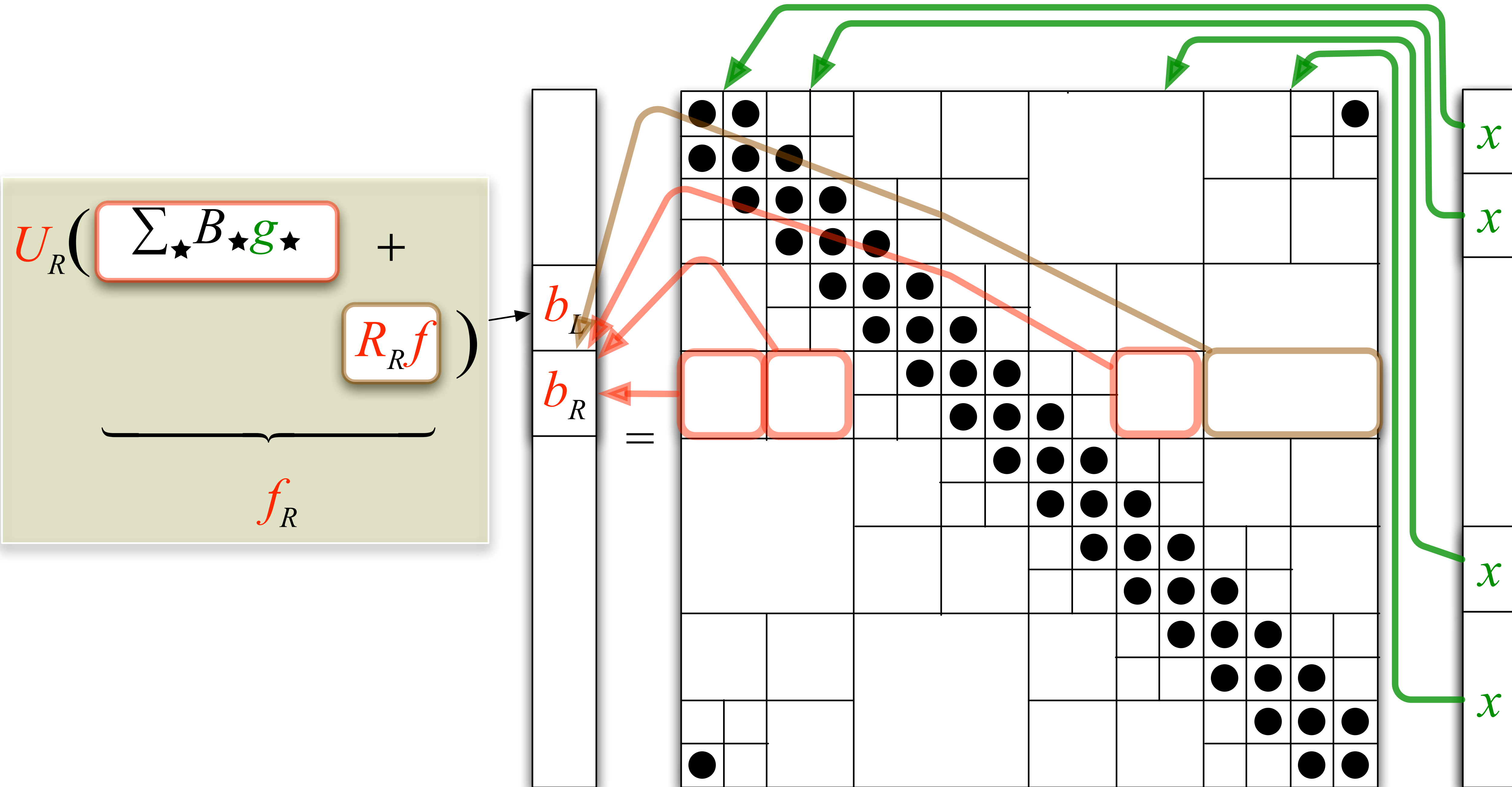
FMM



FMM



FMM



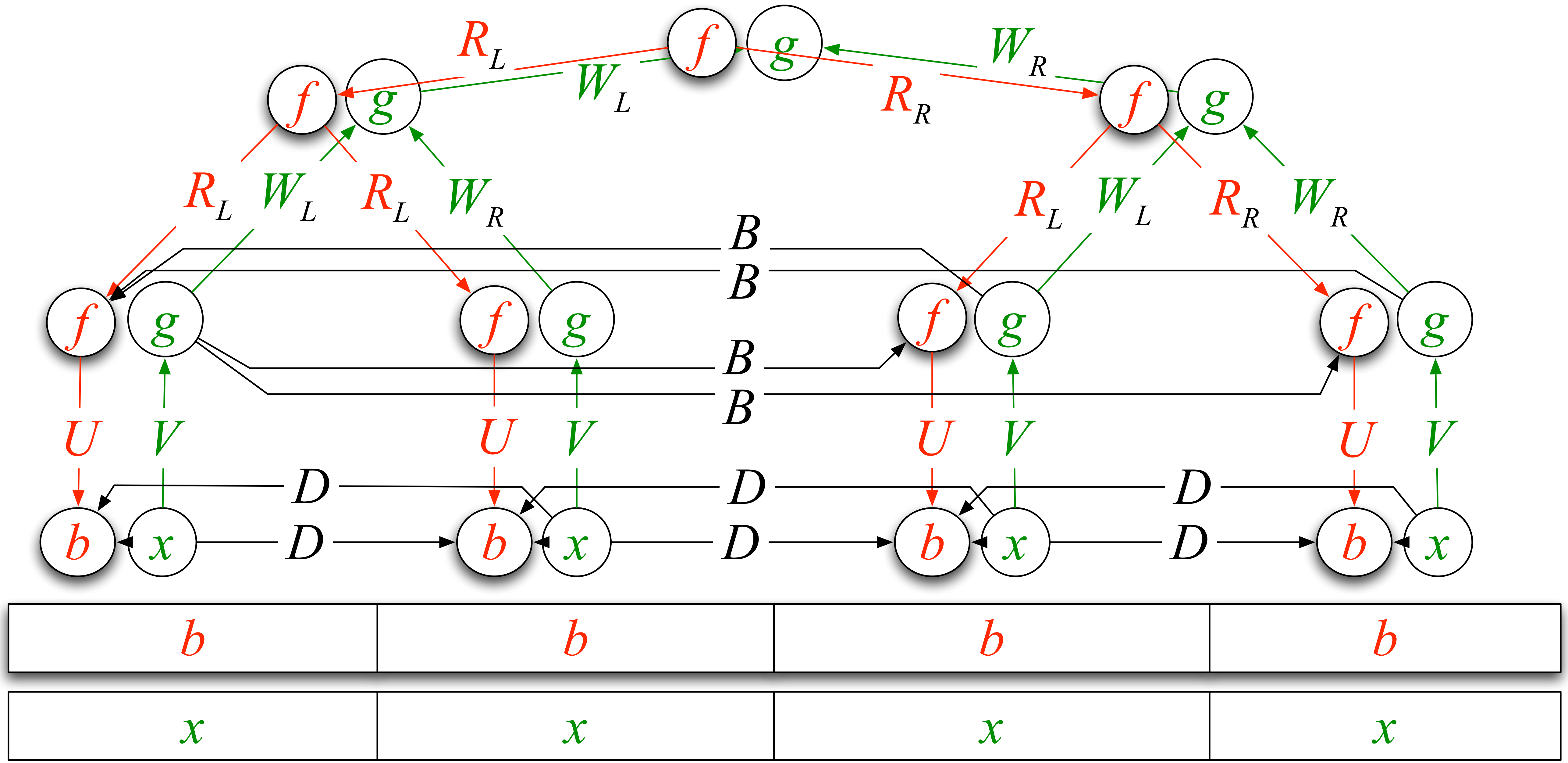
FMM Down Sweep Recursions

$$\boxed{f_L} = \boxed{\sum_\star B_\star g_\star} + \boxed{R_L}\,\boxed{f}$$

$$\boxed{f_R} = \boxed{\sum_\star B_\star g_\star} + \boxed{R_R}\,\boxed{f}$$

$$\boxed{b} = \boxed{\sum_\star D_\star x_\star} + \boxed{U}\,\boxed{f}$$

FMM Signal Flow Graph

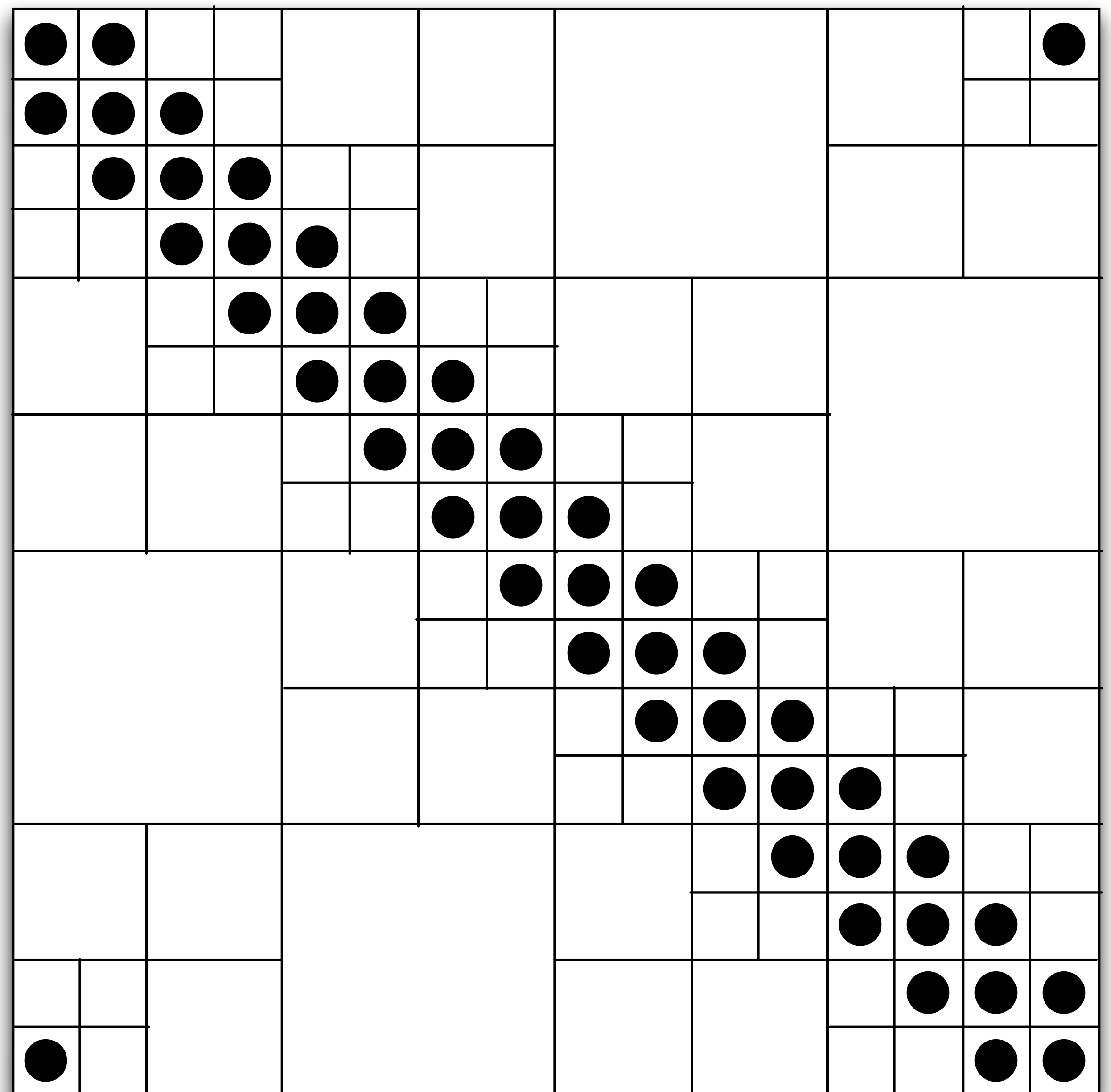


Fast
Solver

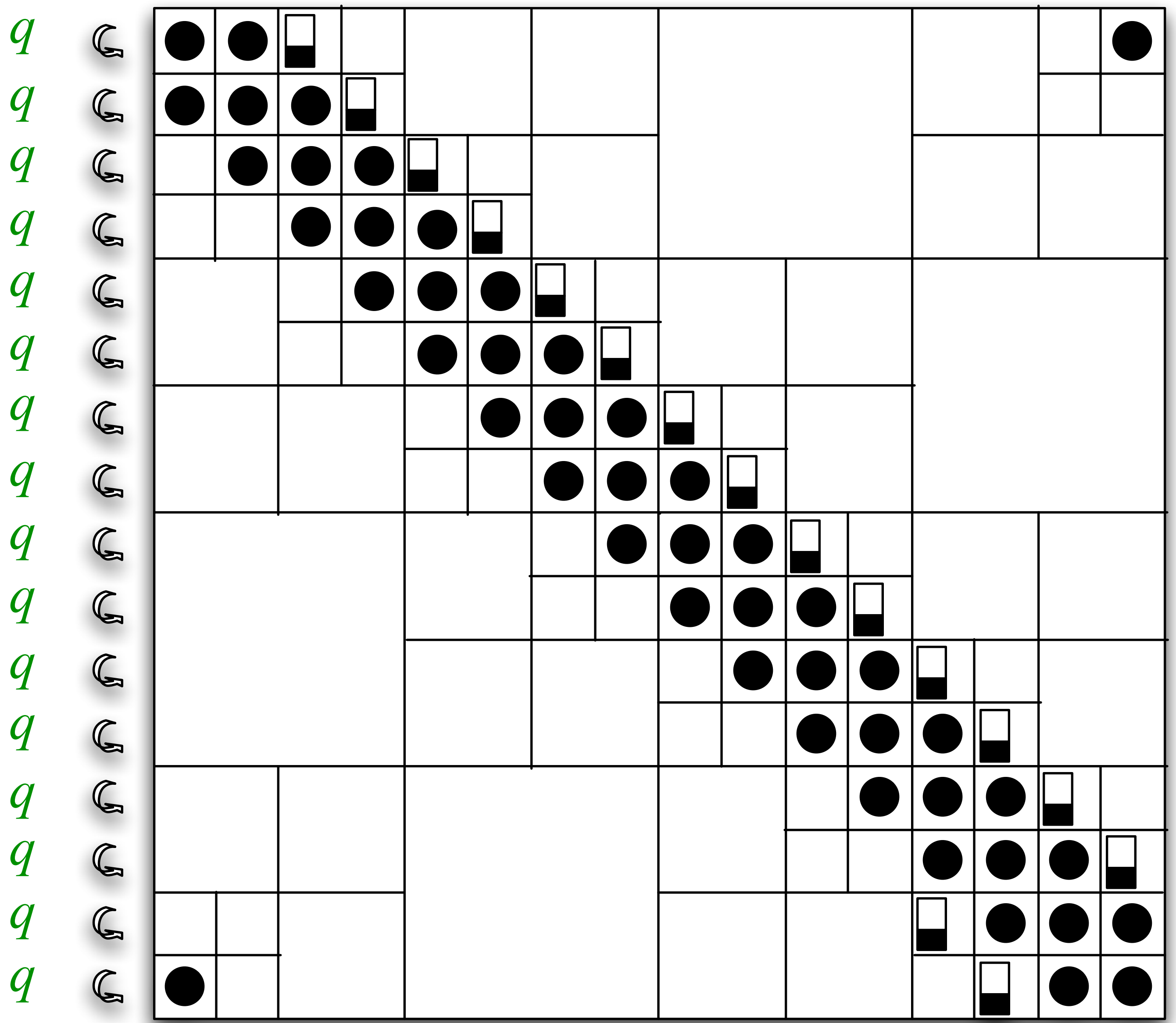
Fast solver

- FMM recursions are *sparse linear* equations in x, b, f, g
- Solved in $O(n^{(1+d)/2})$ flops by standard sparse solvers
- Nested dissection ordering must be used
- Both sparse QR and LU factorization will work
- Too inefficient if $d = 3$

Super-fast solver : Is an $O(n)$ solver possible?

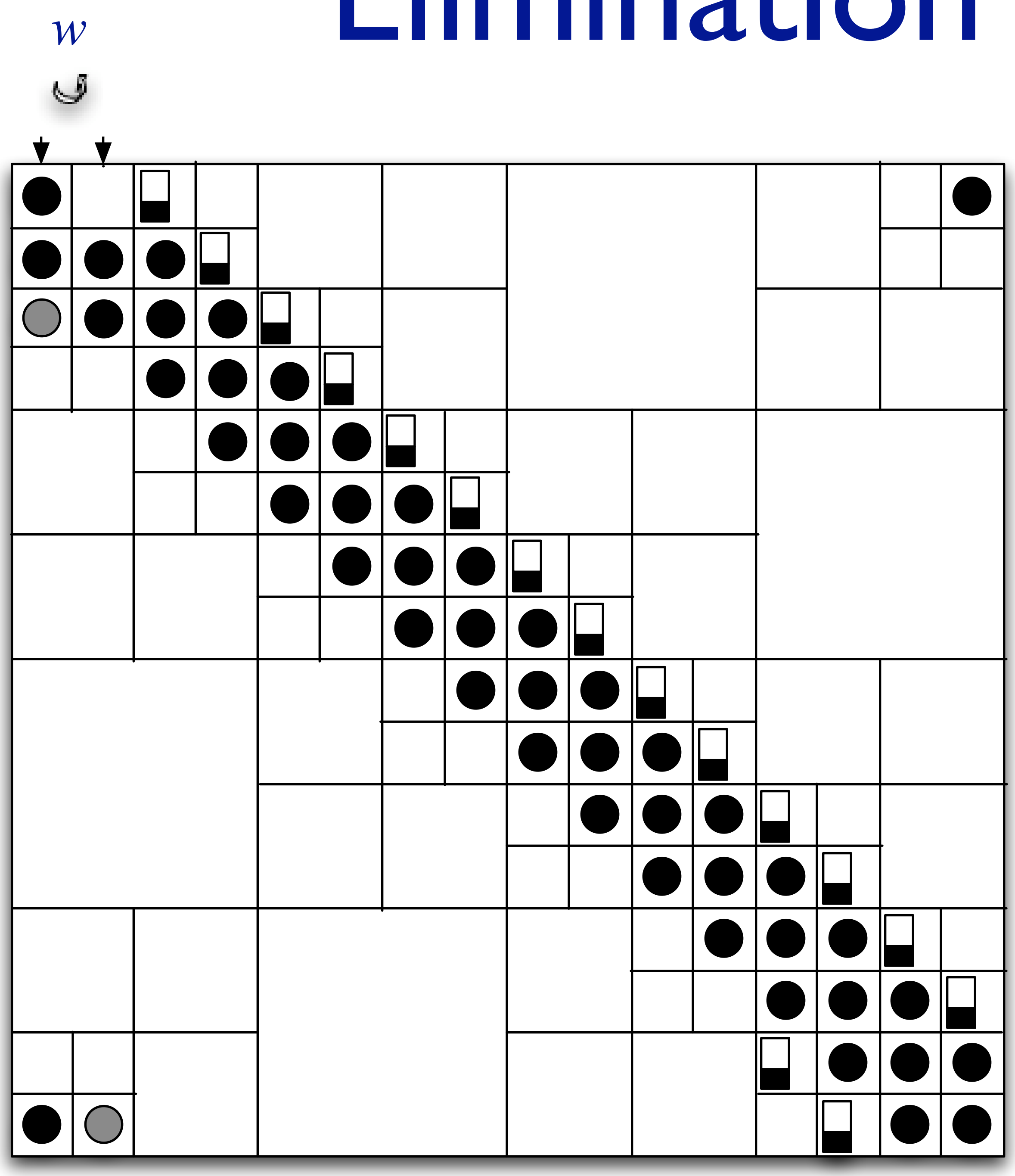


Compression

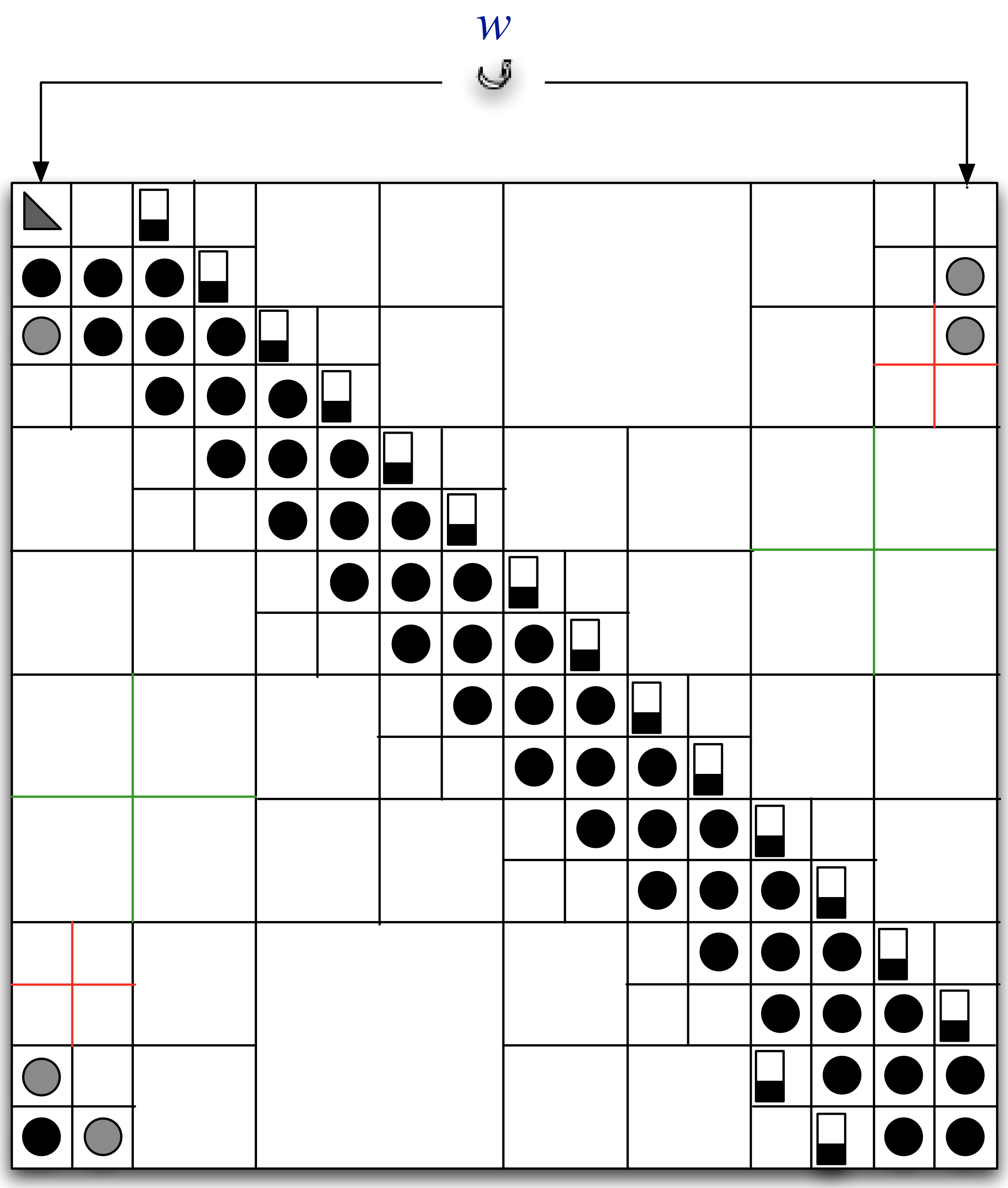


$$q^H q = q q^H = I$$

Elimination

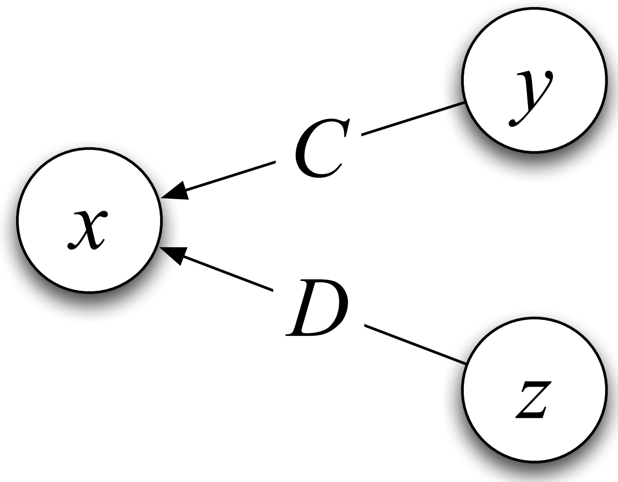


$$w^H w = w w^H = I$$



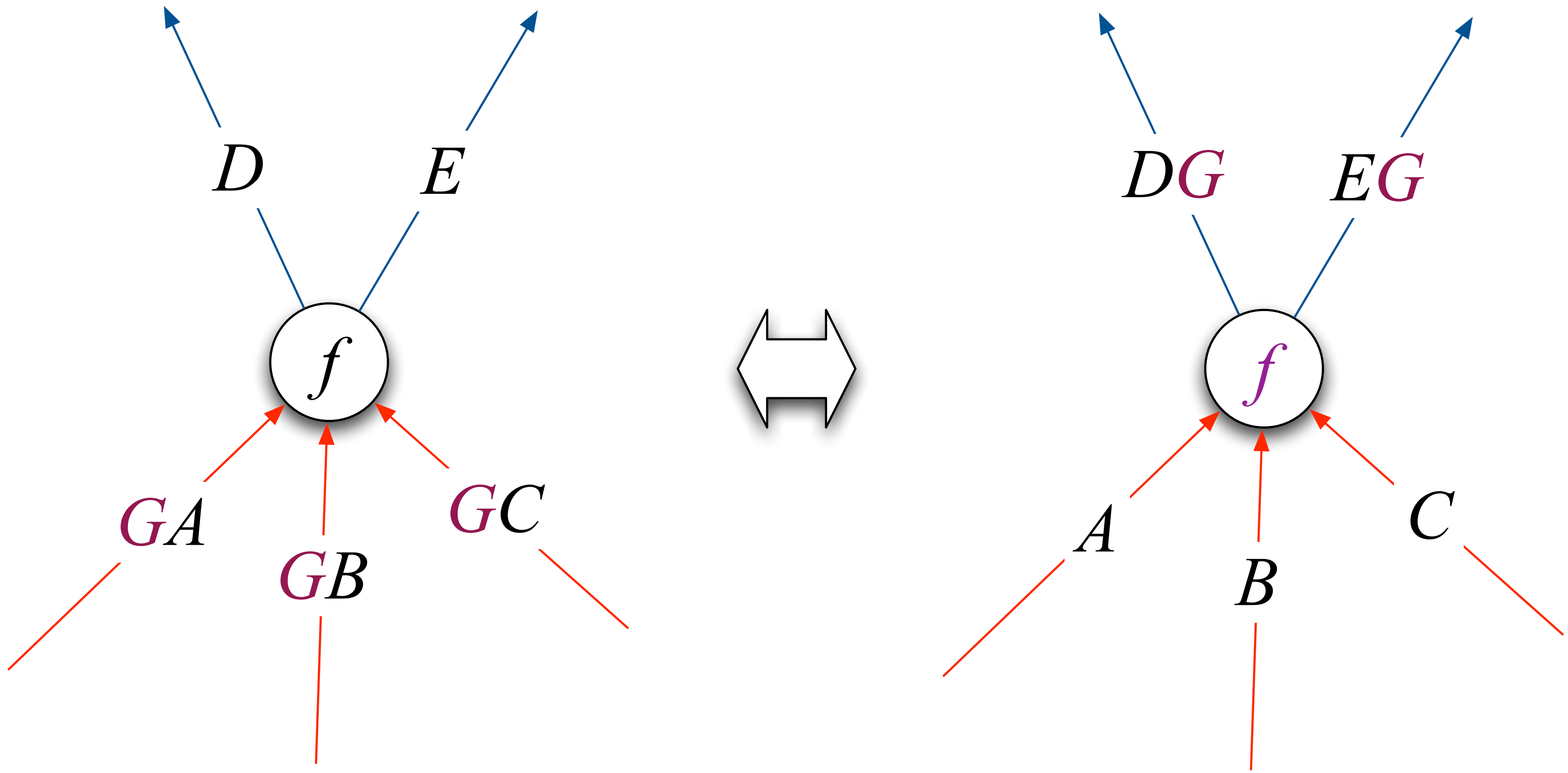
$$w^H w = w w^H = I$$

Elementary Graph Moves

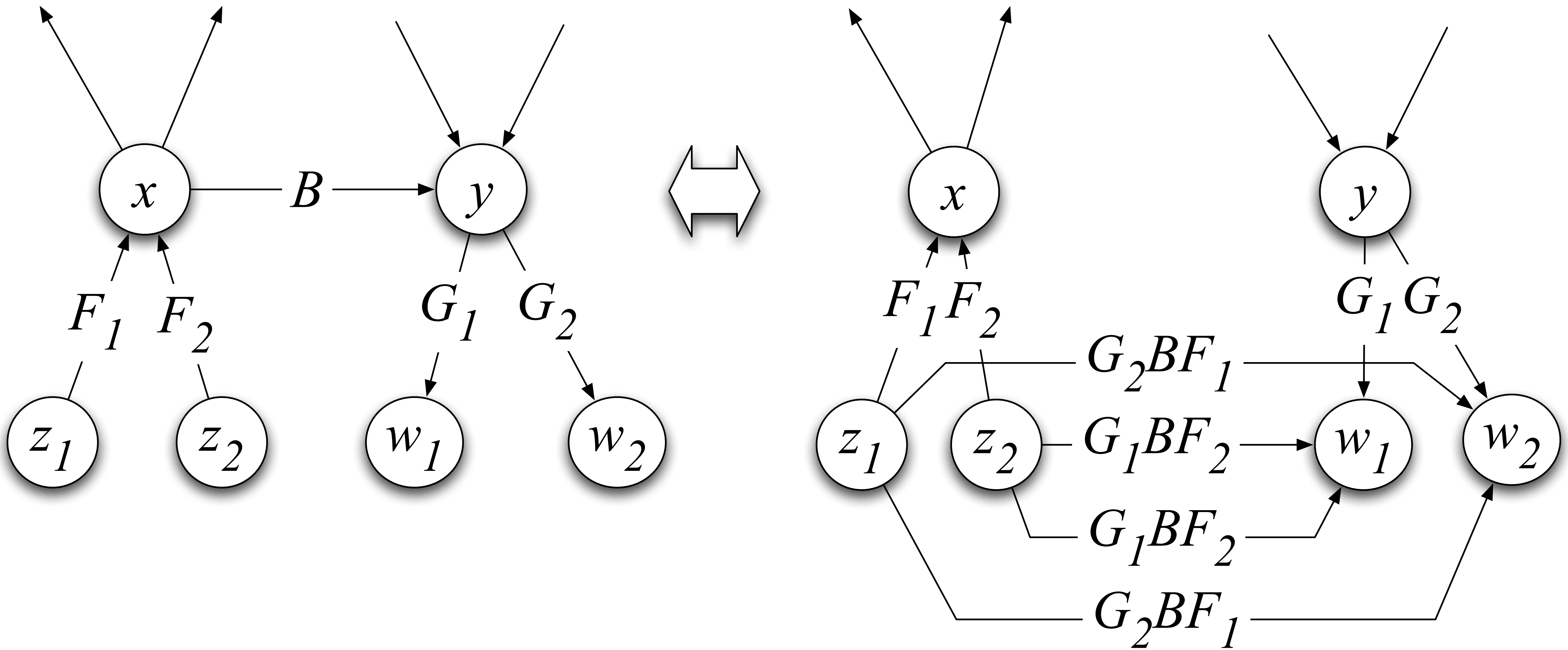


$$x = Cy + Dz$$

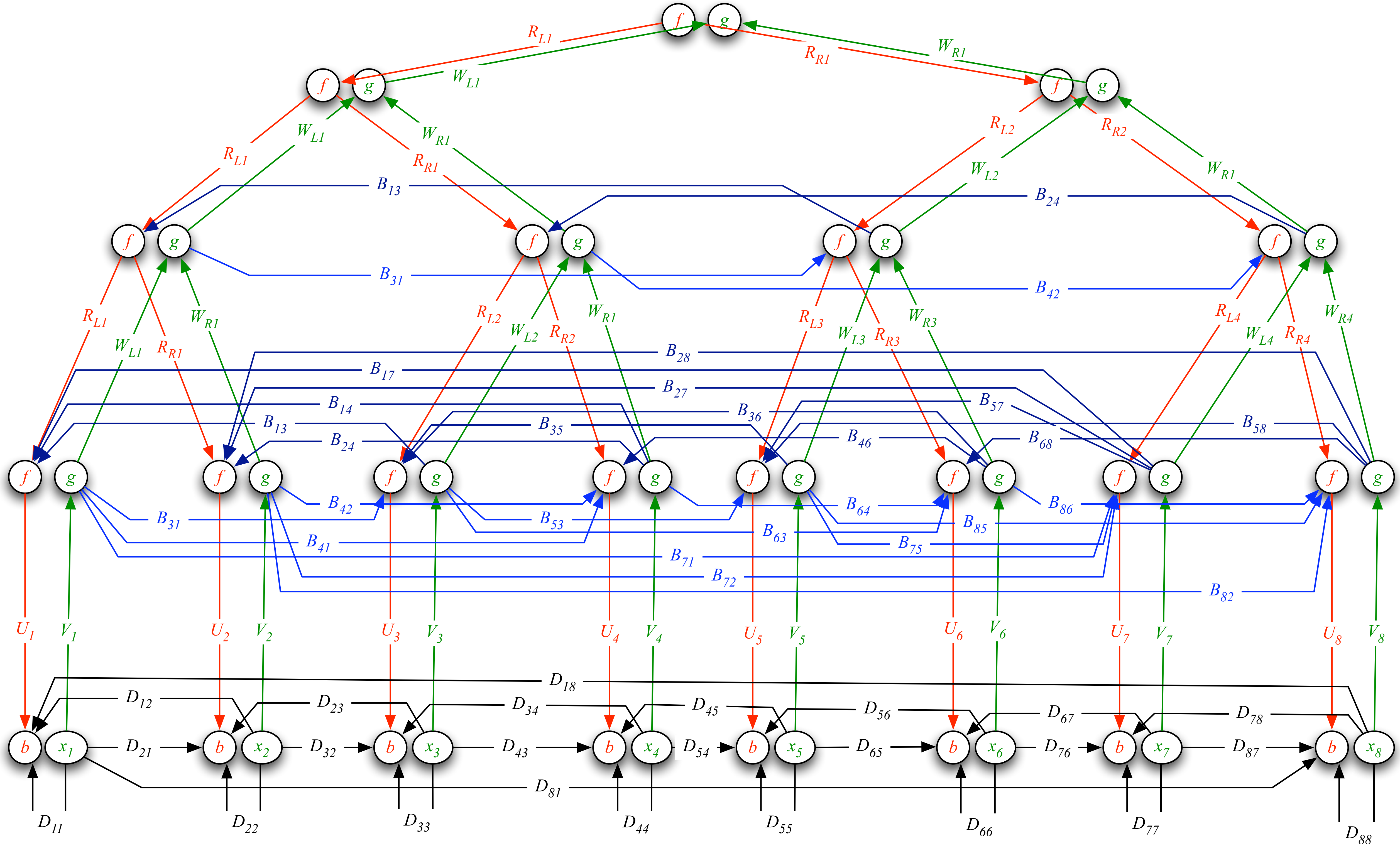
Common Input/Output Factorization



Edge Slide



x
B
y
F_1
F_2
G_1
G_2
z_1
z_2
w_1
w_2
x
y
F_1
F_2
G_1
G_2
G_2BF_1
G_1BF_2
G_1BF_2
G_2BF_1
z_1
z_2
w_1
w_2



f
g
R_L1
W_R1
R_R1
W_L1
R_L2
R_R2
W_L2
W_R1
B_13
B_24
B_31
B_42
R_L1
W_L1
R_R1
W_R1
R_L2
W_L2
R_R2
W_R1
R_L3
W_L3
R_R3
W_R3
R_L4
W_L4
R_R4
W_R4
B_28
B_17
B_27
B_14
B_57
B_36
B_13
B_35
B_58
B_24
B_46
B_68
B_42
B_64
B_86
B_31
B_53
B_85
B_63
B_41
B_75
B_71
B_72
B_82
U_1
V_1
U_2
V_2
U_3
V_3
U_4
V_4
U_5
V_5
U_6
V_6
U_7
V_7
U_8
V_8
D_18
D_12
D_23
D_34
D_45
D_56
D_67
D_78
b
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8
D_21
D_32
D_43
D_54
D_65
D_76
D_87
D_81
D_11
D_22
D_33
D_44
D_55
D_66
D_77
D_88