A new technique for the numerical solution of PDEs*

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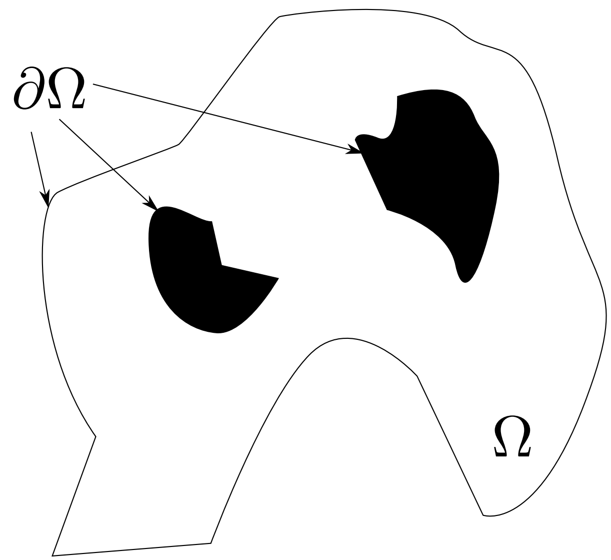
The Problem

Given a connected region Ω with boundary $\partial\Omega$



find (numerically) a vector-valued function u such that

$$\begin{aligned} \mathcal{D}_1(\boldsymbol{x}, u) &= f(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \mathcal{D}_2(\boldsymbol{x}, u) &= g(\boldsymbol{x}), & \boldsymbol{x} \in \partial\Omega. \end{aligned}$$

Assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are local, linear, differential operators.

Current Methods

- Finite difference
- Finite element
- Integral equation methods

Current Difficulties

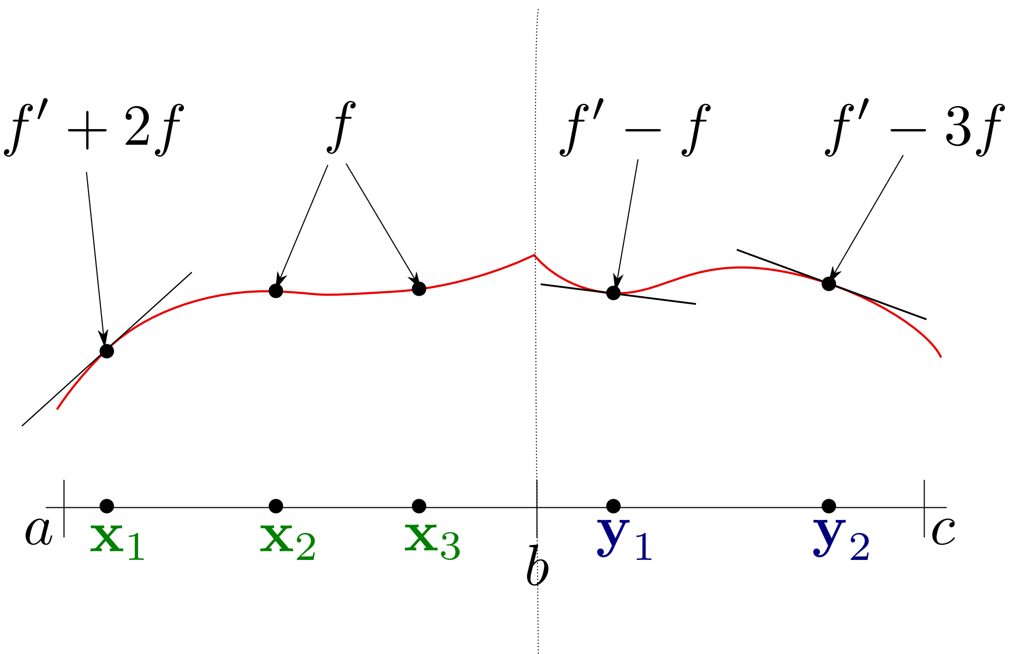
- First-order problems
- Fourth and higher order problems
- Singular (mainly exterior) PDEs
- Curved geometries

Bottleneck

- Memory

A 1D problem

Find piece-wise continuous function that fits given data:



- Data is linear combination of local values of f
- Function is very smooth in patches
- Converge rapidly to underlying function as data increases

On each patch represent f in a (say monomial) basis

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \alpha_n x^n, & a \leqslant x \leqslant b, \\
f(y) &= \sum_{n=0}^{\infty} \beta_n y^n, & b \leqslant y \leqslant c.
\end{aligned}
$$

Constraints on unknowns α_n, β_n and $f(b)$:

$$
\begin{pmatrix}
x_1^{0\prime} + 2x_1^0 & x_1^{1\prime} + 2x_1^1 & \cdots \\
x_2^0 & x_2^1 & \cdots \\
x_3^0 & x_3^1 & \cdots \\
b^0 & b^1 & \cdots
\end{pmatrix}
\begin{pmatrix} \vdots \\ \alpha_n \\ \vdots \end{pmatrix}
=
\begin{pmatrix}
f'(x_1) + 2f(x_1) \\
f(x_2) \\
f(x_3) \\
f(b)
\end{pmatrix}
$$

$$
\begin{pmatrix}
y_1^{0\prime} - y_1^0 & y_1^{1\prime} - 3y_1^1 & \cdots \\
y_2^{0\prime} - y_2^0 & y_2^{1\prime} - 3y_2^1 & \cdots \\
b^0 & b^1 & \cdots
\end{pmatrix}
\begin{pmatrix} \vdots \\ \beta_n \\ \vdots \end{pmatrix}
=
\begin{pmatrix}
f'(y_1) - f(y_1) \\
f'(y_2) - 3f(y_2) \\
f(b)
\end{pmatrix}
$$

- 7 equations and $\infty + \infty + 1$ unknowns
  - Pick 7 basic variables (FD, FEM)
  - Pick **minimum norm** solution (Golomb–Weinberger)

Strategy

- Pick a good basis
- Pick a good norm to minimize
- Memory efficient algorithm to solve equations

Pitfalls

- Accurate evaluation of high-order derivative is difficult
- High-order convergence implies smaller number of equations but higher condition numbers
- Curved geometries can spoil high-order convergence

Chebyshev Basis 1D

$$T_n(x) = \cos(n \cos^{-1} x), \qquad n \in \mathbb{N}, \qquad -1 \leqslant x \leqslant 1.$$

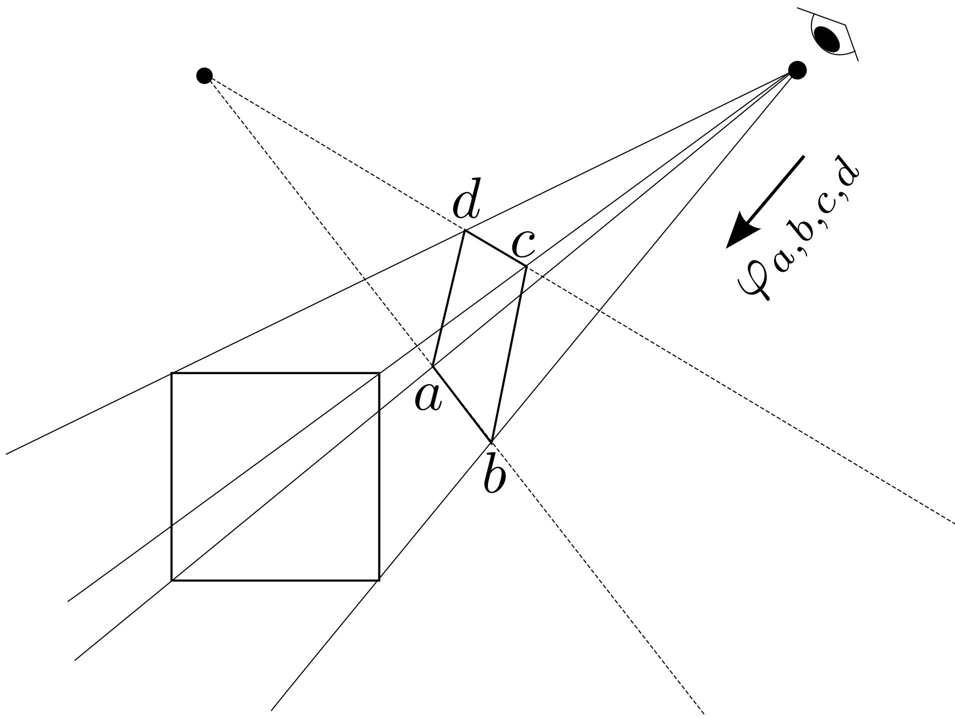
On $[a, b]$ basis is $T_n \circ \varphi_{a,b}$ where $\varphi_{a,b}\colon [a, b] \to [-1, 1]$ is **affine-linear**.

- Under $\cos\theta = x$ this is Fourier cosine basis
- Simple formulas for differentiation and integration
- Two orthogonality conditions

Chebyshev Basis 2D

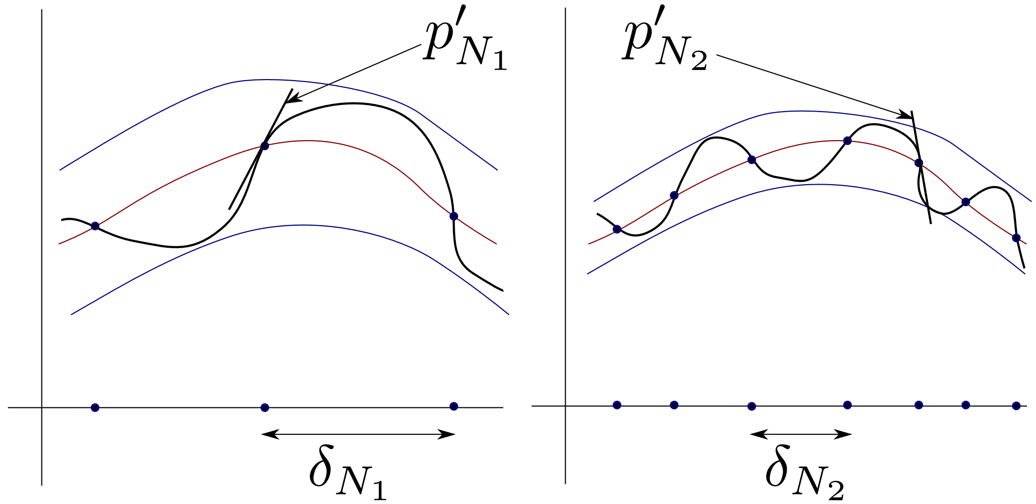
$$T_{n,m}(x,y) = T_n(x)\,T_m(y), \qquad n,m \in \mathbb{N}, \qquad (x,y) \in [-1,1]^2.$$

On convex quadrilaterals basis is $T_{n,m} \circ \varphi_{a,b,c,d}$ where $\varphi_{a,b,c,d}$ is the **perspective transform** from convex quadrilateral $[a,b,c,d]$ to $[-1,1]^2$.



- Simple formulas for $\varphi_{a,b,c,d}$, $\varphi^{-1}_{a,b,c,d}$, $D\varphi_{a,b,c,d}$, $D\varphi^{-1}_{a,b,c,d}$.

Let p_N be the approximant when we have N pieces of data.



If we have interpolation somewhere

$$\|p_N - f\|_\infty \lesssim \|p'_N\|_\infty \delta_N$$

Convergence as $N \to \infty$ if

- $\delta_N \to 0$
- $\|p'_N\|_\infty < \infty$

So minimize $\|p'\|$ when solving PDE!

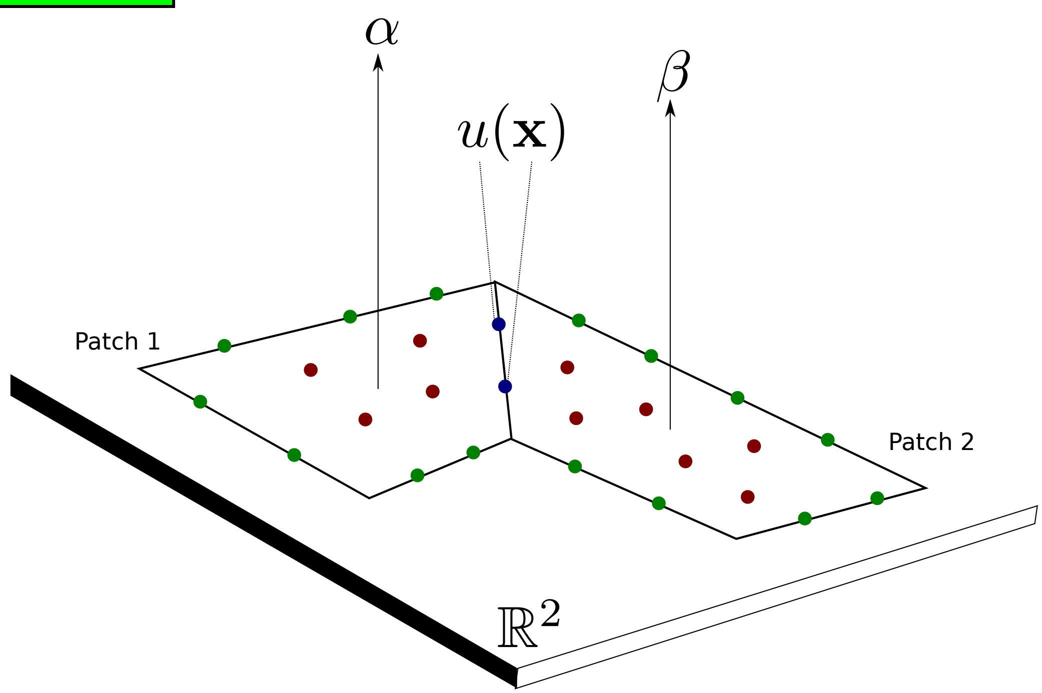
Sobolev norm

- Controlling higher derivatives gives higher order of convergence
- Minimizing $\|p^{(s)}\|_2$ is numerically difficult (thin-plate splines)

Trick: consider the problem "on the torus" instead

$$\begin{aligned} p\big(\varphi_{a,b}^{-1}(\cos\theta)\big) &= \sum_{n=0}^{\infty} \alpha_n \cos(n\theta) \\ \big\|\big(p\circ\varphi_{a,b}^{-1}\circ\cos\big)^{(s)}\big\|_2^2 &= \sum_{n=0}^{\infty} \alpha_n^2\, n^{2s} \end{aligned}$$

- Diagonally weighted 2-norm
- Hough–Vavasis algorithm used to handle ill-conditioning
- Pick $s \simeq 10$ for high-order convergence



α – Chebyshev coefficients of patch 1

β – Chebyshev coefficients of patch 2

$u(\boldsymbol{x})$ – Unknown function value on interface points

● – Interior PDE points

● – Boundary PDE points

● – Interface points

The equations

- At interior points of patch p

$$\left(\mathcal{D}_1(\boldsymbol{x}_i, T_j) \right) \left(\alpha_{p,j} \right) = \left(f(\boldsymbol{x}_i) \right)$$

- At boundary points of patch p

$$\left(\mathcal{D}_2(\boldsymbol{x}_i, T_j) \right) \left(\alpha_{p,j} \right) = \left(g(\boldsymbol{x}_i) \right)$$

- At interface points of patch p

$$\left(T_j(\boldsymbol{x}_i) \quad -I \right) \left(\alpha_{p,j} \quad u(\boldsymbol{x}_i) \right) = \left(0 \right)$$

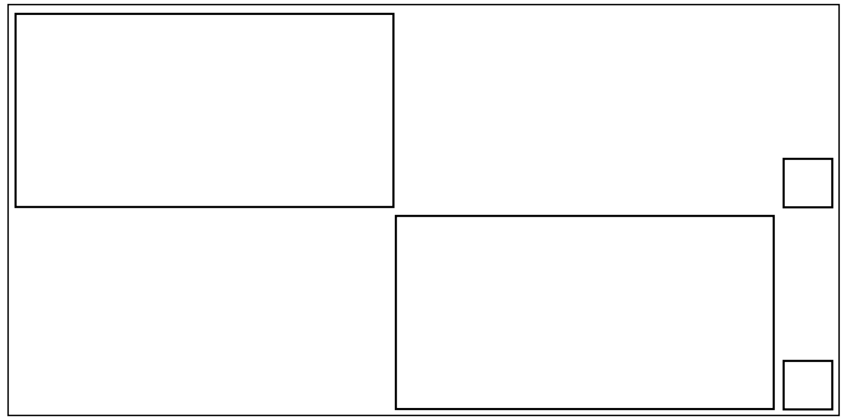
Assembled equations (scaled by $D_s = \text{diag}(\,(1+n)^s\,)$ to bring in Sobolev norm)

$$\begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & 0 & -I \\ 0 & A_{32} & 0 \\ 0 & A_{42} & -I \end{pmatrix} \begin{pmatrix} D_s^{-1} & & \\ & D_s^{-1} & \\ & & I \end{pmatrix} \begin{pmatrix} D_s\alpha \\ D_s\beta \\ u(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \text{fg}_1 \\ 0 \\ \text{fg}_2 \\ 0 \end{pmatrix}.$$

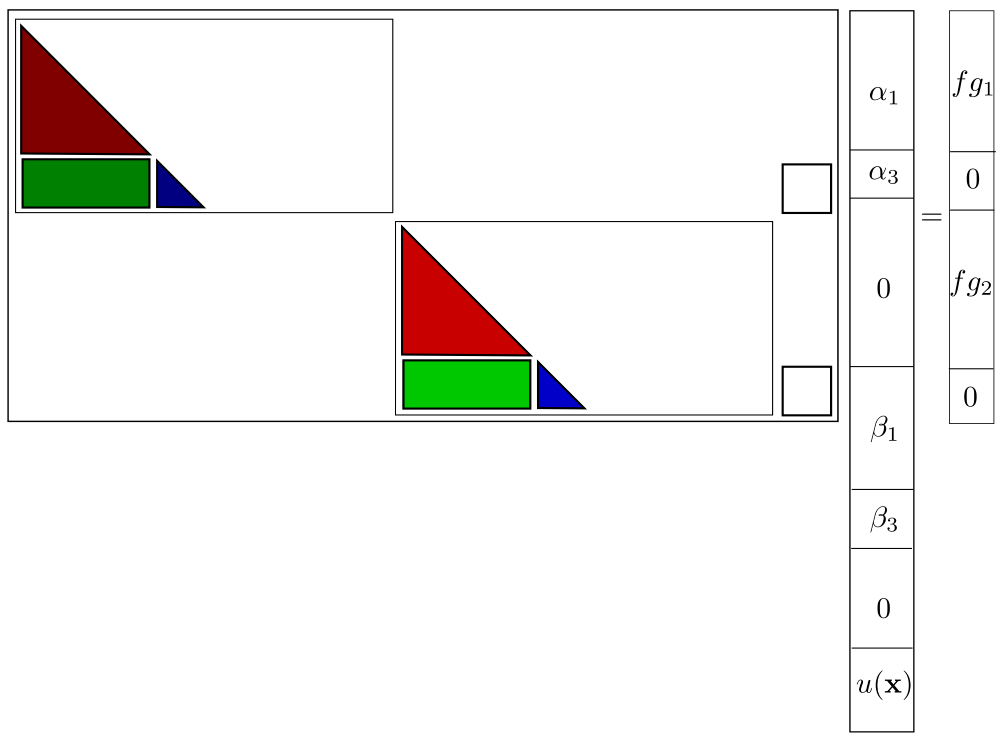
Matrix is fat: **M**inimum **S**obolev **N**orm solution

Equations | Graphical solution

Shape of the coefficient matrix:

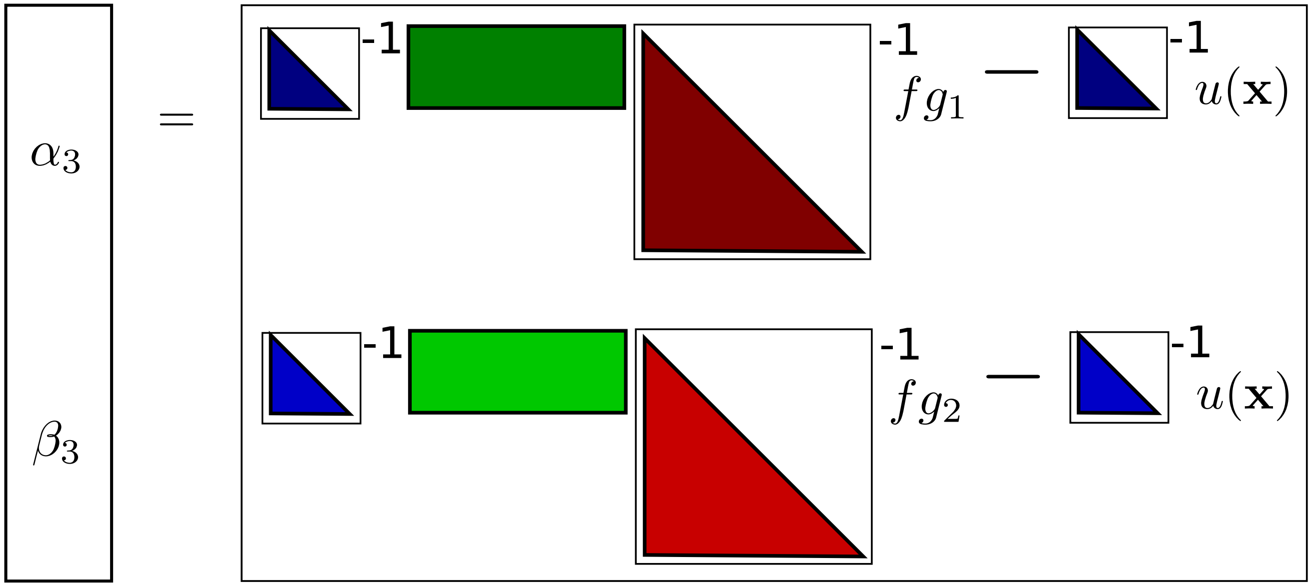


After orthogonal elimination



Equations Reduced

Remaining unknowns



Minimum Sobolev norm solution obtained from sparse least-squares for $u(\boldsymbol{x})$

- Large $s \Rightarrow$ High-order convergence $\Rightarrow$ ill-conditioned $A_{11}D_s^{-1}$
  - Numerical factorization in stages
- Hough–Vavasis type $U_1 \begin{pmatrix} L_1 & 0 \end{pmatrix} V_1^T = A_{11}D_s^{-1}$
  - Poor-man's version
  - SVD $A_{11}D_s^{-1} = U_1\Sigma_1 Q_1^T$
  - LQ factorization $U_1^T A_{11} D_s^{-1} = \Sigma_1 Q_1^T = \begin{pmatrix} L_1 & 0 \end{pmatrix} V_1^T$
- If $V_1^T D_s \alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}^T$ then $\alpha_1 = L_1^{-1} U_1^T \mathrm{fg}_1$
- If $A_{21}D_s^{-1}V_1 = \begin{pmatrix} B_1 & B_2 \end{pmatrix}$, compute $\begin{pmatrix} L_2 & 0 \end{pmatrix} Q_2 = B_2$
- $Q_2\alpha_2 = \begin{pmatrix} \alpha_3 & 0 \end{pmatrix}$ (need MSN solution)
- $\alpha_3 = L_2^{-1} B_1 L_1^{-1} U_1^T \mathrm{fg}_1 - L_2^{-1} u(\boldsymbol{x})$
- Minimum $\|\alpha_3\|^2 + \|\beta_3\|^2$ from sparse least-squares problem

$$\min_{u(\boldsymbol{x})} \left\| \begin{pmatrix} L_2^{-1} \\ \text{patch } 2 \end{pmatrix} \begin{pmatrix} u(\boldsymbol{x}) \end{pmatrix} - \begin{pmatrix} L_2^{-1} B_1 L_1^{-1} U_1^T \mathrm{fg}_1 \\ \text{patch } 2 \end{pmatrix} \right\|_2$$

- Return $u(\boldsymbol{x})$ only on interfaces
- Solution on any patch can be re-computed independently

PDE formulation

- Numerical higher-order derivatives difficult to compute
- Prefer first-order system formulation

On Ω

$$A_1 \frac{\partial w}{\partial x} + A_2 \frac{\partial w}{\partial y} + B\,w = f$$

On $\partial\Omega$

$$C\,w = g$$

With

$$\begin{aligned}
w &: \overline{\Omega} \mapsto \mathbb{R}^p \\
A_1, A_2, B &: \Omega \mapsto \mathbb{R}^{q\times p} \\
f &: \Omega \mapsto \mathbb{R}^q \\
r &: \partial\Omega \mapsto \mathbb{N} \\
C(\boldsymbol{x}) &\in \mathbb{R}^{r(\boldsymbol{x})\times p}, & \boldsymbol{x} \in \partial\Omega \\
g(\boldsymbol{x}) &\in \mathbb{R}^{r(\boldsymbol{x})}, & \boldsymbol{x} \in \partial\Omega
\end{aligned}$$

$$\nabla^T A \nabla u + b^T A \nabla u + c\, u = f$$

where $\nabla = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix}^T$

- New vector unknown:

$$w = \begin{pmatrix} u \\ A\nabla u \end{pmatrix}$$

- First-order system (3×3):

$$\begin{pmatrix} 0 & 1 & 0 \\ A_{11} & 0 & 0 \\ A_{21} & 0 & 0 \end{pmatrix} \frac{\partial w}{\partial x} + \begin{pmatrix} 0 & 0 & 1 \\ A_{12} & 0 & 0 \\ A_{22} & 0 & 0 \end{pmatrix} \frac{\partial w}{\partial y} + \begin{pmatrix} c & b_1 & b_2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} w = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}$$

- First-order system (4×3) with $A = I$:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{\partial w}{\partial x} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \frac{\partial w}{\partial y} + \begin{pmatrix} c & b_1 & b_2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} w = \begin{pmatrix} f \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
\begin{aligned}
\text{Dirichlet} &: \; C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\
\text{Neumann} &: \; C = \begin{pmatrix} 0 & \nu_1 & \nu_2 \end{pmatrix} \\
\text{Tangential} &: \; C = \begin{pmatrix} 0 & \tau_1 & \tau_2 \end{pmatrix} \\
\text{Mixed} &: \; C = \begin{pmatrix} a & b & c \end{pmatrix}
\end{aligned}
$$

$$\nabla^T \nabla u + k^2 u = f$$

In 3×3 first-order form

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial w}{\partial x} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \frac{\partial w}{\partial y} + \begin{pmatrix} k^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} w = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}$$

with

$$w = \begin{pmatrix} u \\ \nabla u \end{pmatrix}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = f$$

In first-order form

$$\begin{pmatrix} x & x^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{\partial w}{\partial x} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\frac{\partial w}{\partial y} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} w = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}$$

with

$$w = \begin{pmatrix} u \\ \nabla u \end{pmatrix}$$

Boundary equations

- Dirichlet: $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$
- Neumann: $C = \begin{pmatrix} 0 & \nu_1 & \nu_2 \end{pmatrix}$

Wave equation (1+1):

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial y^2} = f$$

First-order form

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{\partial w}{\partial x} + \begin{pmatrix} 0 & 0 & -\frac{1}{c^2} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\frac{\partial w}{\partial y} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} w = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}$$

with

$$w = \begin{pmatrix} u \\ \nabla u \end{pmatrix}$$

Boundary equations:

- $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ at initial time
- $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ at physical end-points
- $C = \begin{pmatrix} \square & \square & \square \end{pmatrix}_{0\times 3}$ at final time

Heat equation (1+1):

$$\kappa\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = f$$

First-order form:

$$\begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix}\frac{\partial w}{\partial x} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\frac{\partial w}{\partial y} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} w = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

with

$$w = \begin{pmatrix} u \\ \frac{\partial u}{\partial x} \end{pmatrix}$$

Boundary equations:

- $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$ at initial time and physical end-points
- $C = \begin{pmatrix} \square & \square \end{pmatrix}_{0\times 2}$ at final time

$$\nabla^T(\beta u) = f$$

My first-order form:

$$\beta_1 \frac{\partial u}{\partial x} + \beta_2 \frac{\partial u}{\partial y} + (\nabla^T \beta) u = f$$

Examples:

- Circular advection: $\beta(x, y) = \begin{pmatrix} -y & x \end{pmatrix}^T / \|(x, y)\|$
- 1+1 advection: $\beta = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$

Boundary equation (ν is normal to boundary)

- If $\beta^T \nu < 0$ then $C = \begin{pmatrix} 1 \end{pmatrix}$
- If $\beta^T \nu > 0$ then $C = \begin{pmatrix} \square \end{pmatrix}_{0 \times 1}$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ D_{11} & 0 & 0 & 0 & 0 \\ D_{12} & 0 & 0 & 0 & 0 \\ 0 & D_{33} & 0 & 0 & 0 \end{pmatrix} \frac{\partial w}{\partial x} + \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & D_{12} & 0 & 0 & 0 \\ 0 & D_{22} & 0 & 0 & 0 \\ D_{33} & 0 & 0 & 0 & 0 \end{pmatrix} \frac{\partial w}{\partial y} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} w = \begin{pmatrix} F_1 \\ F_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where

$$w = \begin{pmatrix} u_1 \\ u_2 \\ D_{11}\frac{\partial u_1}{\partial x} + D_{12}\frac{\partial u_2}{\partial y} \\ D_{12}\frac{\partial u_1}{\partial x} + D_{22}\frac{\partial u_2}{\partial y} \\ D_{33}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \end{pmatrix}$$

Boundary equation:

- Displacement condition: $C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$
- Traction condition: $C = \begin{pmatrix} 0 & 0 & \nu_1 & 0 & \nu_2 \\ 0 & 0 & 0 & \nu_2 & \nu_1 \end{pmatrix}$

where $\nu = \begin{pmatrix} \nu_1 & \nu_2 \end{pmatrix}^T$ is the normal to the boundary.

FYI: $D_{i,j}$ depend on elastic material parameters (e.g., Young's modulus and Poisson's ratio)

PDE formulation | Linearized stationary Navier–Stokes

$$
\begin{aligned}
-\nabla p + \nu \nabla^T \nabla u + (b^T \nabla) u + (u^T \nabla) b &= f_1 \\
\nabla^T u &= f_2
\end{aligned}
$$

In first-order form

$$
\begin{pmatrix}
b_1 & 0 & -1 & \nu & 0 & 0 & 0 \\
0 & b_1 & 0 & 0 & 0 & \nu & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \frac{\partial w}{\partial x} +
\begin{pmatrix}
b_2 & 0 & 0 & 0 & \nu & 0 & 0 \\
0 & b_2 & -1 & 0 & 0 & 0 & \nu \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \frac{\partial w}{\partial y} +
$$

$$
\begin{pmatrix}
\frac{\partial b_1}{\partial x} & \frac{\partial b_1}{\partial y} & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial b_2}{\partial x} & \frac{\partial b_2}{\partial y} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} w =
\begin{pmatrix}
f_{1,1} \\ f_{1,2} \\ f_2 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
$$

Unknowns

$$w = \begin{pmatrix} u \\ p \\ \nabla u_1 \\ \nabla u_2 \end{pmatrix}$$

Boundary equations:

- Flow on walls: $C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

- Pressure on inlets and outlets: $C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

PDE formulation | Bi-harmonic

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f$$

Bi-harmonic in first-order form

$$\begin{pmatrix} 0&0&0&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}\frac{\partial w}{\partial x}+\begin{pmatrix} 0&0&0&0&0&0&0&2&1\\ 0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0 \end{pmatrix}\frac{\partial w}{\partial y} \quad +$$

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0\\ 0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&-1 \end{pmatrix} w \quad = \quad \begin{pmatrix} f\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}$$

Unknowns

$$w = \begin{pmatrix} u(x) & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x\, y} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^3 u}{\partial x^3} & \frac{\partial^3 u}{\partial x^2 y} & \frac{\partial^3 u}{\partial y^3} \end{pmatrix}^T$$

Boundary equations

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu_1 & \nu_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

PDE formulation | Div–Curl

$$\nabla^T A\, u \;=\; \rho \qquad \text{(div)}$$

$$\nabla^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u \;=\; \omega \qquad \text{(curl)}$$

In first-order form

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & 1 \end{pmatrix} \frac{\partial u}{\partial x} + \begin{pmatrix} A_{21} & A_{22} \\ -1 & 0 \end{pmatrix} \frac{\partial u}{\partial y} + \begin{pmatrix} \frac{\partial A_{11}}{\partial x} + \frac{\partial A_{21}}{\partial y} & \frac{\partial A_{12}}{\partial x} + \frac{\partial A_{22}}{\partial y} \\ 0 & 0 \end{pmatrix} u = \begin{pmatrix} \rho \\ \omega \end{pmatrix}$$

Boundary equations (well–posed: Auchmuty–Alexander)

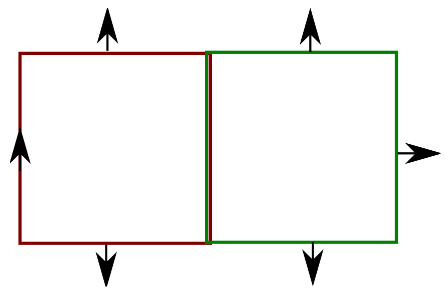
- $\partial\Omega = \Gamma_N \cup \Gamma_T$ and $\Gamma_N \cap \Gamma_T = \emptyset$ and Γ_N, Γ_T are connected non-empty sets
- $C = \nu^T A$ on Γ_N
- $C = \tau^T$ on Γ_T

If A is dis-continuous, the interface continuity conditions become:

$$\begin{pmatrix} \nu^T A_+ \\ \tau^T \end{pmatrix} u_+ = \begin{pmatrix} \nu^T A_- \\ \tau^T \end{pmatrix} u_- = v$$

- Code outputs this common value on the interface
- Can be modified to make it inhomogenous

Constant coeff. div-curl on:



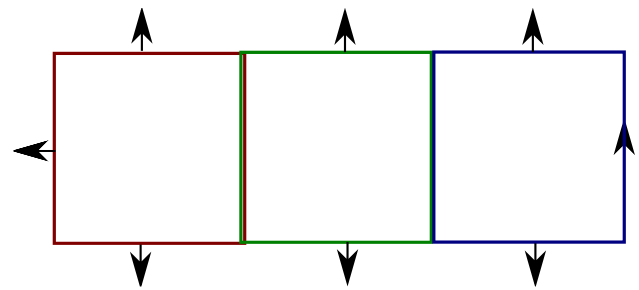
Arrows: normal or tangential boundary condition

$$u = \begin{pmatrix} (1+x^2+y^2)^{-1} \\ x^2 - 2y^2 + x\,y - x + 1 \end{pmatrix}$$

Grid spacing $h \simeq 0.08$

Accuracy: about 10 digits

Constant coeff. div-curl on:



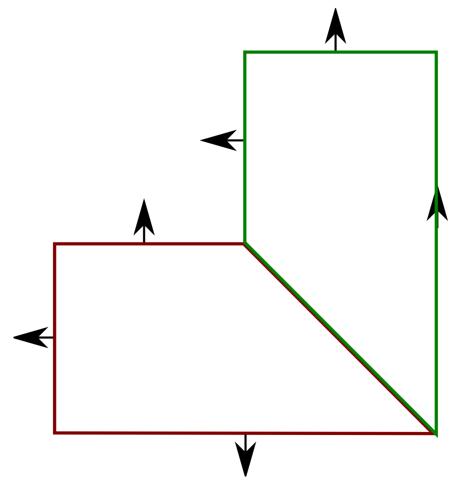
Arrows: normal or tangential boundary condition

$$u = \begin{pmatrix} (1+x^2+y^2)^{-1} \\ x^2 - 2y^2 + x\,y - x + 1 \end{pmatrix}$$

Grid spacing $h \simeq 0.08$

Accuracy: about 10 digits

Constant coeff. div-curl on:



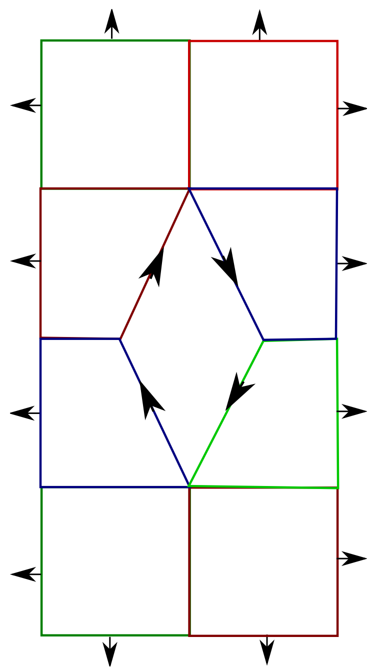
Arrows: normal or tangential boundary condition

$$u = \begin{pmatrix} (1+x^2+y^2)^{-1} \\ x^2 - 2y^2 + x\,y - x + 1 \end{pmatrix}$$

Grid spacing $h \simeq 0.1$

Accuracy: about 8 digits

Constant coeff. div-curl on:



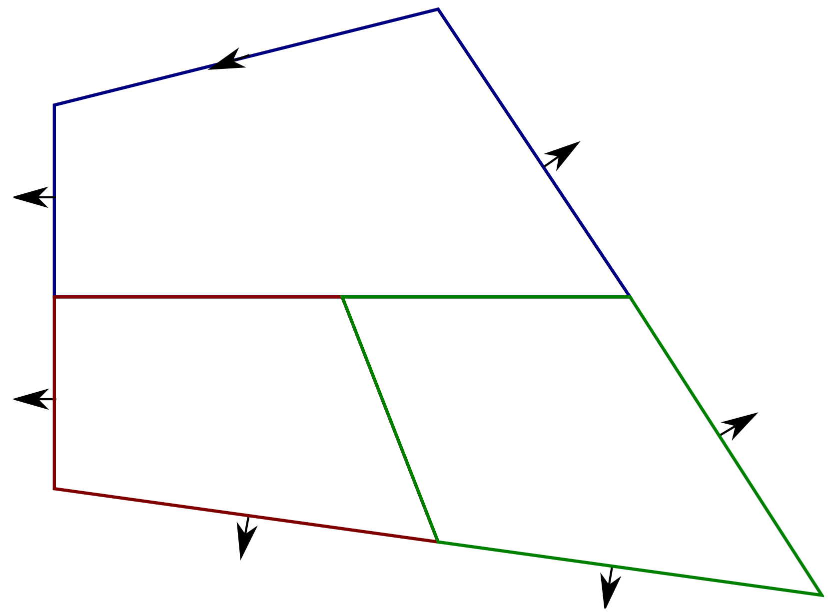
Arrows: normal or tangential boundary condition

$$u = \begin{pmatrix} (1 + x^2 + y^2)^{-1} \\ x^2 - 2y^2 + x\,y - x + 1 \end{pmatrix}$$

Grid spacing $h \simeq 0.08$

Accuracy: about 11 digits

Constant coeff. div-curl on:



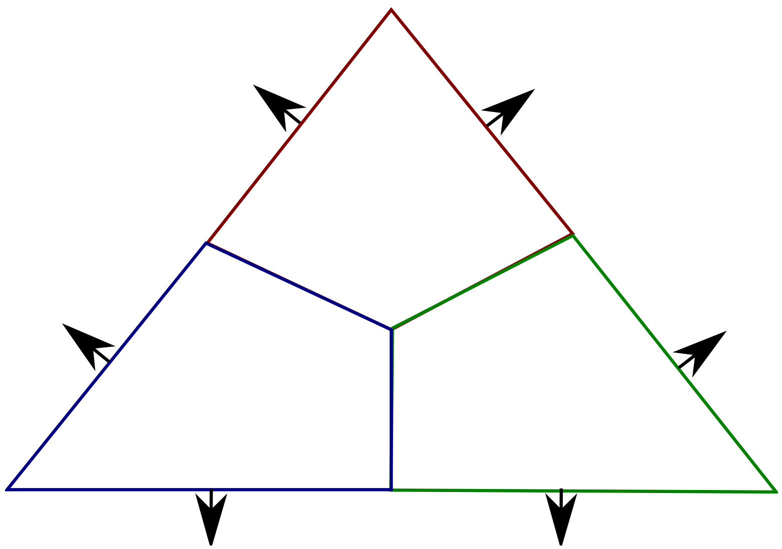
Arrows: normal or tangential boundary condition

$$u = \begin{pmatrix} (1+x^2+y^2)^{-1} \\ x^2 - 2y^2 + x\,y - x + 1 \end{pmatrix}$$

Grid spacing $h \simeq 0.3$

Accuracy: about 8 digits

Constant coeff. div-curl on:



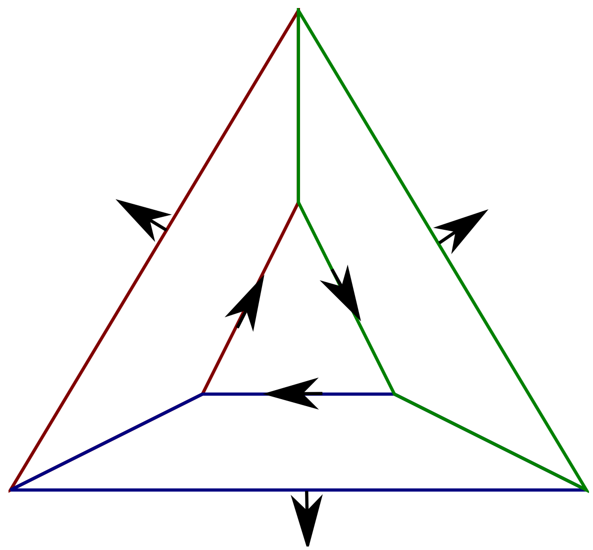
Only normal boundary condition!

$$u = \begin{pmatrix} (1+x^2+y^2)^{-1} \\ x^2 - 2y^2 + x\,y - x + 1 \end{pmatrix}$$

Grid spacing $h \simeq 0.3$

Accuracy: about 6 digits

Constant coeff. div-curl on:



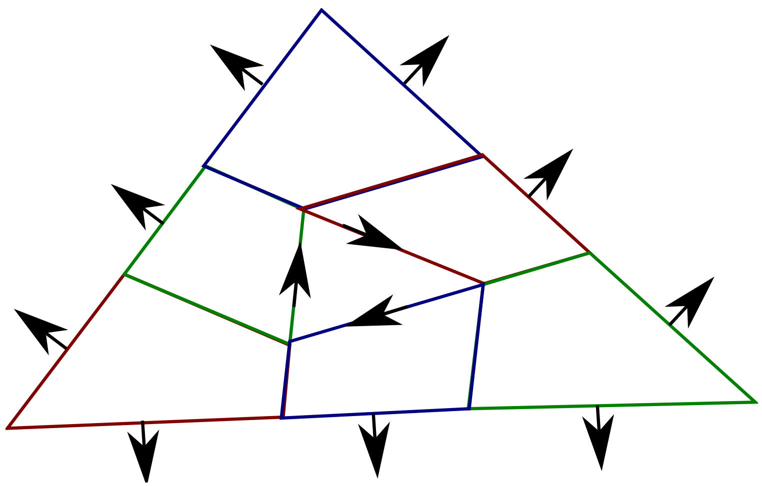
Arrows: normal or tangential boundary condition

$$u = \begin{pmatrix} (1 + x^2 + y^2)^{-1} \\ x^2 - 2y^2 + x\,y - x + 1 \end{pmatrix}$$

Grid spacing $h \simeq 0.5$

Accuracy: about 6 digits

Constant coeff. div-curl on:



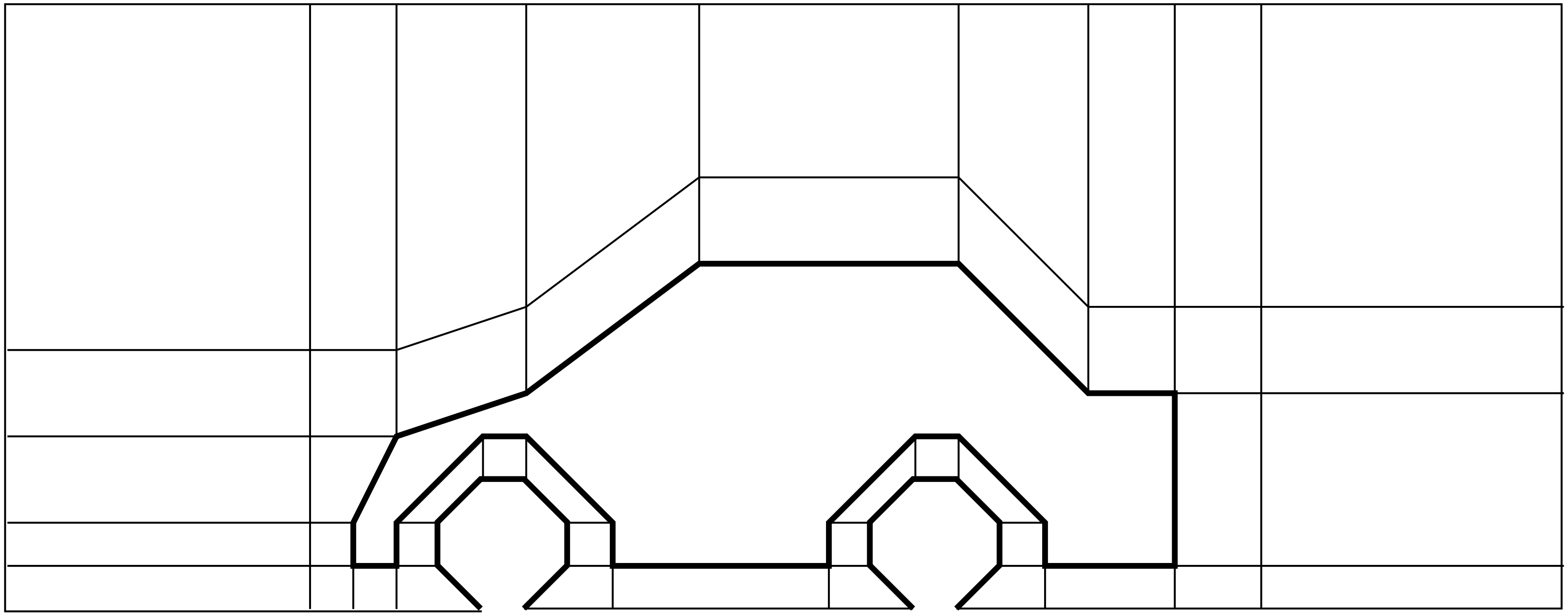
Arrows: normal or tangential boundary condition

$$u = \begin{pmatrix} (1+x^2+y^2)^{-1} \\ x^2 - 2y^2 + x\,y - x + 1 \end{pmatrix}$$

Grid spacing $h \simeq 0.1$

Accuracy: about 8 digits

Constant coeff. div-curl on:



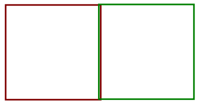
Normal boundary condition on car boundary

Tangential boundary condition on outside boundary

$$u = \begin{pmatrix} (1+x^2+y^2)^{-1} \\ x^2 - 2y^2 + x\,y - x + 1 \end{pmatrix}$$

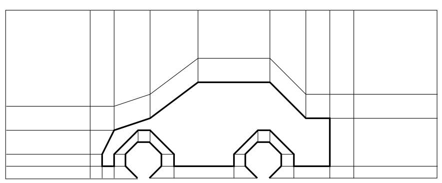
Grid spacing $h \simeq 0.6$

Accuracy: about 9 digits

■ Domain: 

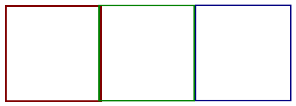
Grid spacing $h \simeq 0.08$

- Accuracy 4×3 formulation: about 8 digits
- Accuracy 3×3 formulation: about 8 digits

■ Domain: 

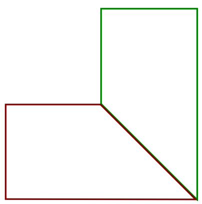
Grid spacing $h \simeq 0.6$

- Accuracy 4×3 formulation: about 4 digits
- Accuracy 3×3 formulation: about 4 digits

■ Domain: 

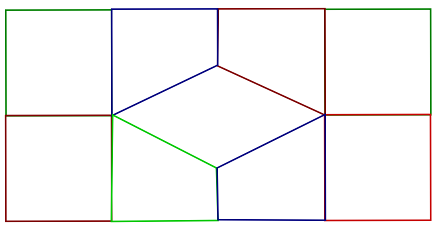
Grid spacing $h \simeq 0.08$

Accuracy: about 8 digits

■ Domain: 

Grid spacing $h \simeq 0.1$

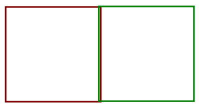
Accuracy about 6 digits

■ Domain: 

Grid spacing $h \simeq 0.08$

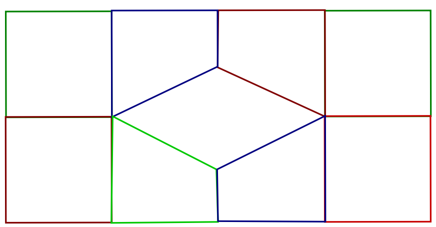
Accuracy about 7 digits

Constant coefficient wave equation

- Domain: 

Grid spacing $h \simeq 0.08$

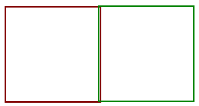
Accuracy about 7 digits

- Domain: 

Grid spacing $h \simeq 0.08$

Accuracy about 7 digits

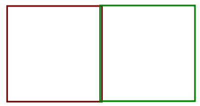
Constant coefficient heat equation

- Domain: 

Grid spacing $h \simeq 0.08$

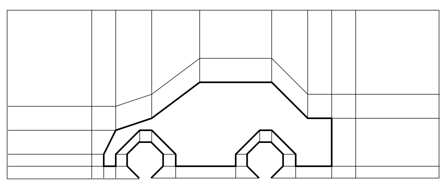
Accuracy about 8 digits

Tests | Bi-harmonic equation

■ Domain: 

Grid spacing $h \simeq 0.08$

Accuracy about 6 digits

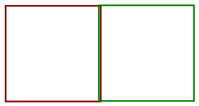
■ Domain: 

Grid spacing $h \simeq 0.6$

Accuracy about 3 digits

Grid spacing $h \simeq 0.4$

Accuracy about 4 digits

■ Domain: 

Grid spacing $h \simeq 0.08$

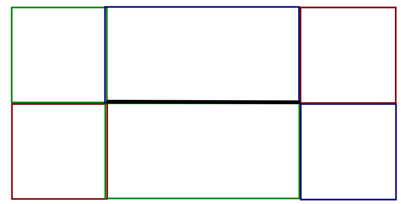
Type: Circulating field

Accuracy about 9 digits

Type: 1+1

Accuracy about 9 digits

- Constant coefficient div-curl
- Domain is $[-2,2]\times[-1,1]$ with a slit on $[-1,1]\times\{0\}$



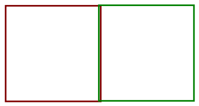
- Grid spacing $h \simeq 0.1$
- Let $v(z) = (z^2-1)^{3/2}$ with branch cut on slit

$$u = \begin{pmatrix} v_{\mathrm{Imag}} \\ v_{\mathrm{Real}} \end{pmatrix}$$

- u_1 is dis-continuous across slit, but u_2 is continuous
- Poles at $(-1,0)$ and $(1,0)$
- Single normal boundary condition on slit
  - Accuracy about 5 digits
- Double tangential boundary condition on slit
  - Accuracy about 4 digits

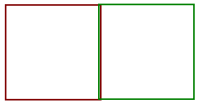
Tests | High-frequency Helmholtz

$$\nabla^T \nabla u + 10^8 u = f$$

- Domain: 

Grid spacing $h \simeq 0.08$

Accuracy about 7 digits

■ Domain: 

Young's modulus: 1

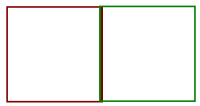
Poisson's ratio: 0.25

Displacement boundary condition on left vertical edge

Grid spacing $h \simeq 0.08$

Accuracy about 9 digits

Tests | Linearized stationary Navier–Stokes

■ Domain: 

Base flow: $\begin{pmatrix} x+y & x\,y \end{pmatrix}$

True solution: $\begin{pmatrix} y(2-y)-1 & (1+x^2+y^2)^{-1} & 1-x \end{pmatrix}$

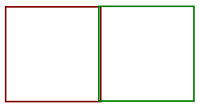
Pressure boundary condition on left and right vertical edge

Flow boundary condition on top and bottom

Grid spacing $h \simeq 0.08$

Accuracy about 6 digits

Type: 3×3 formulation

■ Domain: 

Singularity on left edge

Dirichlet boundary conditions

Grid spacing $h \simeq 0.08$

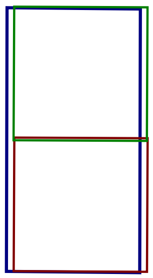
- Solution $(1 + x^2 + y^2)^{-1}$

  Accuracy about 8 digits

- Solution $\text{Real}\big(z^{5/2}\big)$

  Accuracy about 6 digits

Type: 3×3 formulation

■ Domain: 

Third patch behind first two patches

Vertical edges are distinct

Solution Real$\left(z^{5/2}\right)$

Top edges share boundary data

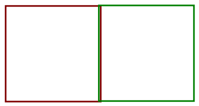
Bottom edges share boundary data (periodic)

Vertical edges do not share boundary data

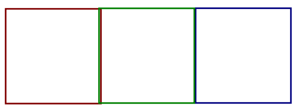
- Grid spacing $h \simeq 0.15$
  Accuracy about 3 digits

- Grid spacing $h \simeq 0.1$
  Accuracy about 5 digits

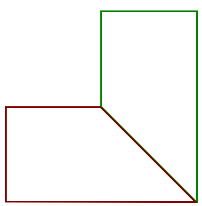
Solution in all cases is suitably modified $(1+x^2+y^2)^{-1}$

■ Domain: 

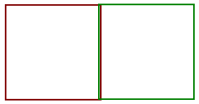
Div–curl coeff. A is 1 on first patch, 2 on second patch
Grid spacing $h \simeq 0.08$
Accuracy about 10 digits

■ Domain: 

Div–curl coeff. A is 1 on first patch, 2 on second patch, 3 on third patch
Grid spacing $h \simeq 0.08$
Accuracy about 10 digits

■ Domain: 

Div–curl coeff. A is 1 on first patch, 2 on second patch
Grid spacing $h \simeq 0.1$
Accuracy about 8 digits

■ Domain: 

Tanential on right edge with value 1

Normal on left edge with value 0

Normal on top edge with value x

Normal on bottom edge with value $-x$

Grid spacing $h_1 \simeq 0.077$ and $h_2 \simeq 0.071$

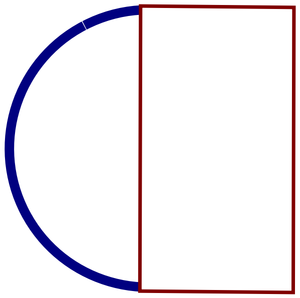
Two solutions agreed to about 4 digits

$$\nabla^T \nabla u - u = f$$

- Solution: $(1 + 10(x - y^2)^2)^{-1}$
  - Singularity is on a parabola

- Domain: Circle of diameter 1

Grid spacing	Error
0.1	2E-3
0.075	3E-4
0.05	4E-5
0.0375	1E-5
0.025	2E-6

Domain: 

Grid spacing	Digits of accuracy
0.4	3
0.2	4
0.1	8

Proof

Quick outline

- Keep number of patches fixed
- Assume infinite order polynomials on each patch (kernel or RBF approach)
- Compactness argument based on uniform bound on Sobolev norm
- Limit is a continuous function that satsifies PDE inside patches and on boundaries
- If PDE theory says this is a classical solution we are done
- With lot more effort we can also look at finite-order polynomials on each patch

Summary

- Golomb–Weinberger MSN technique for PDEs
- First-order formulation

Future Work

- Realistic tests with domain specialists
  - Send us your 2D problem
  - Use our code
- Eigenvalue problems
- Nonlinear problems
- 3D

Thank you!