A new technique for the numerical solution of PDEs*

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The Problem

Given a connected region $\Omega$ with boundary $\partial \Omega$

find (numerically) a vector-valued function $u$ such that

\[
D_1(x, u) = f(x), \quad x \in \Omega,
\]
\[
D_2(x, u) = g(x), \quad x \in \partial \Omega.
\]

Assume that $D_1$ and $D_2$ are local, linear, differential operators.
**Current Methods**

- Finite difference
- Finite element
- Integral equation methods

**Current Difficulties**

- First-order problems
- Fourth and higher order problems
- Singular (mainly exterior) PDEs
- Curved geometries

**Bottleneck**

- Memory
A 1D problem

Find piece-wise continuous function that fits given data:

- Data is linear combination of local values of $f$
- Function is very smooth in patches
- Converge rapidly to underlying function as data increases
A 1D problem

On each patch represent $f$ in a (say monomial) basis

$$f(x) = \sum_{n=0}^{\infty} \alpha_n x^n, \quad a \leq x \leq b,$$

$$f(y) = \sum_{n=0}^{\infty} \beta_n y^n, \quad b \leq y \leq c.$$ 

Constraints on unknowns $\alpha_n, \beta_n$ and $f(b)$:

$$\begin{pmatrix} x_1' + 2x_1^0 & x_1^1 + 2x_1^1 & \cdots \\ x_2^0 & x_2^1 & \cdots \\ x_3^0 & x_3^1 & \cdots \\ b^0 & b^1 & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ \alpha_n \\ \vdots \end{pmatrix} = \begin{pmatrix} f'(x_1) + 2f(x_1) \\ f(x_2) \\ f(x_3) \\ f(b) \end{pmatrix}$$

$$\begin{pmatrix} y_1' - y_1^0 & y_1^1 - 3y_1^1 & \cdots \\ y_2' - y_2^0 & y_2^1 - 3y_2^1 & \cdots \\ y_3' - y_3^0 & y_3^1 - 3y_3^1 & \cdots \\ b^0 & b^1 & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ \beta_n \\ \vdots \end{pmatrix} = \begin{pmatrix} f'(y_1) - f(y_1) \\ f'(y_2) - 3f(y_2) \\ f(b) \end{pmatrix}$$

- 7 equations and $\infty + \infty + 1$ unknowns
  - Pick 7 basic variables (FD, FEM)
  - Pick **minimum norm** solution (Golomb–Weinberger)
**Strategy**

- Pick a good basis
- Pick a good norm to minimize
- Memory efficient algorithm to solve equations

**Pitfalls**

- Accurate evaluation of high-order derivative is difficult
- High-order convergence implies smaller number of equations but higher condition numbers
- Curved geometries can spoil high-order convergence
Chebyshev Basis \hspace{1cm} 1D

\[ T_n(x) = \cos(n \cos^{-1} x), \quad n \in \mathbb{N}, \quad -1 \leq x \leq 1. \]

On \([a, b]\) basis is \(T_n \circ \varphi_{a,b}\) where \(\varphi_{a,b} : [a, b] \rightarrow [-1, 1]\) is affine-linear.

- Under \(\cos \theta = x\) this is Fourier cosine basis
- Simple formulas for differentiation and integration
- Two orthogonality conditions
Chebyshev Basis 2D

\[ T_{n,m}(x, y) = T_n(x) T_m(y), \quad n, m \in \mathbb{N}, \quad (x, y) \in [-1, 1]^2. \]

On convex quadrilaterals basis is \( T_{n,m} \circ \varphi_{a,b,c,d} \) where \( \varphi_{a,b,c,d} \) is the perspective transform from convex quadrilateral \([a, b, c, d]\) to \([-1, 1]^2\).

- Simple formulas for \( \varphi_{a,b,c,d}, \varphi_{a,b,c,d}^{-1}, D\varphi_{a,b,c,d}, D\varphi_{a,b,c,d}^{-1} \).
Let $p_N$ be the approximant when we have $N$ pieces of data.

If we have interpolation somewhere

$$\|p_N - f\|_{\infty} \lesssim \|p'_N\|_{\infty} \delta_N$$

Convergence as $N \to \infty$ if

- $\delta_N \to 0$
- $\|p'_N\|_{\infty} < \infty$

So minimize $\|p'\|$ when solving PDE!
- Controlling higher derivatives gives higher order of convergence

- Minimizing $\|p^{(s)}\|_2$ is numerically difficult (thin-plate splines)

**Trick:** consider the problem “on the torus” instead

\[
p(\varphi_{a,b}^{-1}(\cos \theta)) = \sum_{n=0}^{\infty} \alpha_n \cos (n \theta)
\]

\[
\| (p \circ \varphi_{a,b}^{-1} \circ \cos )^{(s)} \|_2^2 = \sum_{n=0}^{\infty} \alpha_n^2 n^{2s}
\]

- Diagonally weighted 2-norm

- Hough–Vavasis algorithm used to handle ill-conditioning

- Pick $s \approx 10$ for high-order convergence
\( \alpha \) – Chebyshev coefficients of patch 1
\( \beta \) – Chebyshev coefficients of patch 2
\( u(x) \) – Unknown function value on interface points

- – Interior PDE points
- – Boundary PDE points
- – Interface points
The equations

- At interior points of patch $p$
  \[
  \left( D_1(x_i, T_j) \right) \left( \alpha_{p,j} \right) = \left( f(x_i) \right)
  \]

- At boundary points of patch $p$
  \[
  \left( D_2(x_i, T_j) \right) \left( \alpha_{p,j} \right) = \left( g(x_i) \right)
  \]

- At interface points of patch $p$
  \[
  \left( T_j(x_i) - I \right) \left( \alpha_{p,j} \ u(x_i) \right) = \left( 0 \right)
  \]

Assembled equations (scaled by $D_s = \text{diag}((1+n)^s)$ to bring in Sobolev norm)

\[
\begin{pmatrix}
A_{11} & 0 & 0 \\
A_{21} & 0 & -I \\
0 & A_{32} & 0 \\
0 & A_{42} & -I
\end{pmatrix}
\begin{pmatrix}
D_{s}^{-1} & \\
D_{s}^{-1} & I
\end{pmatrix}
\begin{pmatrix}
D_s \alpha \\
D_s \beta \\
u(x)
\end{pmatrix}
= \begin{pmatrix}
fg_1 \\
0 \\
fg_2 \\
0
\end{pmatrix}.
\]

Matrix is fat: Minimum Sobolev Norm solution
Shape of the coefficient matrix:

After orthogonal elimination

\[
\begin{align*}
\alpha_1 & \quad f g_1 \\
\alpha_3 & \quad 0 \\
\beta_1 & \quad f g_2 \\
\beta_3 & \quad 0 \\
u(x) & \quad 0
\end{align*}
\]
Remaining unknowns

\[
\begin{bmatrix}
\alpha_3 \\
\beta_3
\end{bmatrix}
= \begin{bmatrix}
\begin{array}{c}
-1 \\
-1
\end{array}
\begin{array}{c}
fg_1 \\
fg_2
\end{array}
-1
\end{bmatrix} u(\mathbf{x})
\]

Minimum Sobolev norm solution obtained from sparse least-squares for \( u(\mathbf{x}) \)
• Large $s \Rightarrow$ High-order convergence $\Rightarrow$ ill-conditioned $A_{11}D_s^{-1}$
  
  – Numerical factorization in stages
• Hough–Vavasis type $U_1 (L_1 \ 0)V_1^T = A_{11}D_s^{-1}$
  
  – Poor-man’s version
  
  – SVD $A_{11}D_s^{-1} = U_1\Sigma_1Q_1^T$
  
  – $LQ$ factorization $U_1^TA_{11}D_s^{-1} = \Sigma_1Q_1^T = (L_1 \ 0)V_1^T$

• If $V_1^TD_s\alpha = (\alpha_1 \ \alpha_2)^T$ then $\alpha_1 = L_1^{-1}U_1^Tfg_1$
• If $A_{21}D_s^{-1}V_1 = (B_1 \ B_2)$, compute $(L_2 \ 0)Q_2 = B_2$
• $Q_2\alpha_2 = (\alpha_3 \ 0)$ (need MSN solution)
• $\alpha_3 = L_2^{-1}B_1L_1^{-1}U_1^Tfg_1 - L_2^{-1}u(x)$
• Minimum $\|\alpha_3\|^2 + \|\beta_3\|^2$ from sparse least-squares problem

$$\min_{u(x)} \left\| \begin{pmatrix} L_2^{-1} \\ \text{patch 2} \end{pmatrix} u(x) - \begin{pmatrix} L_2^{-1}B_1L_1^{-1}U_1^Tfg_1 \\ \text{patch 2} \end{pmatrix} \right\|_2$$

• Return $u(x)$ only on interfaces
• Solution on any patch can be re-computed independently
PDE formulation

- Numerical higher-order derivatives difficult to compute
- Prefer first-order system formulation

On $\Omega$

\[ A_1 \frac{\partial w}{\partial x} + A_2 \frac{\partial w}{\partial y} + B w = f \]

On $\partial \Omega$

\[ C w = g \]

With

\[ w : \bar{\Omega} \rightarrow \mathbb{R}^p \]
\[ A_1, A_2, B : \Omega \rightarrow \mathbb{R}^{q \times p} \]
\[ f : \Omega \rightarrow \mathbb{R}^q \]
\[ r : \partial \Omega \rightarrow \mathbb{N} \]
\[ C(\mathbf{x}) \in \mathbb{R}^{r(\mathbf{x}) \times p}, \quad \mathbf{x} \in \partial \Omega \]
\[ g(\mathbf{x}) \in \mathbb{R}^{r(\mathbf{x})}, \quad \mathbf{x} \in \partial \Omega \]
\[ \nabla^T A \nabla u + b^T A \nabla u + c u = f \]

where \( \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})^T \)

- New vector unknown:
  \[ w = \begin{pmatrix} u \\ A \nabla u \end{pmatrix} \]

- First-order system \((3 \times 3)\):
  \[
  \begin{pmatrix}
    0 & 1 & 0 \\
    A_{11} & 0 & 0 \\
    A_{21} & 0 & 0
  \end{pmatrix}
  \frac{\partial w}{\partial x} + \begin{pmatrix}
    0 & 0 & 1 \\
    A_{12} & 0 & 0 \\
    A_{22} & 0 & 0
  \end{pmatrix}
  \frac{\partial w}{\partial y} + \begin{pmatrix}
    c & b_1 & b_2 \\
    0 & -1 & 0 \\
    0 & 0 & -1
  \end{pmatrix}
  w = \begin{pmatrix}
    f \\
    0 \\
    0
  \end{pmatrix}
  \]

- First-order system \((4 \times 3)\) with \(A = I\):
  \[
  \begin{pmatrix}
    0 & 1 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 1
  \end{pmatrix}
  \frac{\partial w}{\partial x} + \begin{pmatrix}
    0 & 0 & 1 \\
    0 & 0 & 0 \\
    1 & 0 & 0 \\
    0 & -1 & 0
  \end{pmatrix}
  \frac{\partial w}{\partial y} + \begin{pmatrix}
    c & b_1 & b_2 \\
    0 & -1 & 0 \\
    0 & 0 & -1 \\
    0 & 0 & 0
  \end{pmatrix}
  w = \begin{pmatrix}
    f \\
    0 \\
    0 \\
    0
  \end{pmatrix}
  \]
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Dirichlet : \( C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \)

Neumann : \( C = \begin{pmatrix} 0 & \nu_1 & \nu_2 \end{pmatrix} \)

Tangential : \( C = \begin{pmatrix} 0 & \tau_1 & \tau_2 \end{pmatrix} \)

Mixed : \( C = \begin{pmatrix} a & b & c \end{pmatrix} \)
\[ \nabla^T \nabla u + k^2 u = f \]

In $3 \times 3$ first-order form

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\frac{\partial w}{\partial x} + \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\frac{\partial w}{\partial y} + \begin{pmatrix}
k^2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
w = \begin{pmatrix}
f \\
0 \\
0
\end{pmatrix}
\]

with

\[
w = \begin{pmatrix}
u \\
\nabla u
\end{pmatrix}
\]
PDE formulation  Poisson in polar coordinates

\[
x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = f
\]

In first-order form

\[
\begin{pmatrix}
x & x^2 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}\frac{\partial w}{\partial x} + \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}\frac{\partial w}{\partial y} + \begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}w = \begin{pmatrix} f \\ 0 \end{pmatrix}
\]

with

\[
w = \begin{pmatrix} u \\ \nabla u \end{pmatrix}
\]

Boundary equations

- Dirichlet: \( C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \)

- Neumann: \( C = \begin{pmatrix} 0 & \nu_1 & \nu_2 \end{pmatrix} \)
Wave equation (1+1):

\[
\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial y^2} = f
\]

First-order form

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \frac{\partial w}{\partial x} + \begin{pmatrix}
0 & 0 & -\frac{1}{c^2} \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} \frac{\partial w}{\partial y} + \begin{pmatrix}
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix} w = \begin{pmatrix}
f \\
0
\end{pmatrix}
\]

with

\[
w = \begin{pmatrix}
u \\
\nabla u
\end{pmatrix}
\]

Boundary equations:

- \( C = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \) at initial time
- \( C = \begin{pmatrix}
1 & 0 & 0
\end{pmatrix} \) at physical end-points
- \( C = (\Box \ \Box \ \Box)_{0 \times 3} \) at final time
Heat equation (1+1):

\[
\kappa \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = f
\]

First-order form:

\[
\begin{pmatrix}
0 & \kappa \\
1 & 0
\end{pmatrix} \frac{\partial w}{\partial x} + \begin{pmatrix}
-1 & 0 \\
0 & 0
\end{pmatrix} \frac{\partial w}{\partial y} + \begin{pmatrix}
0 & 0 \\
0 & -1
\end{pmatrix} w = \begin{pmatrix}
f \\
0
\end{pmatrix}
\]

with

\[
w = \begin{pmatrix}
u \\
\frac{\partial u}{\partial x}
\end{pmatrix}
\]

Boundary equations:

- \( C = (1 \ 0) \) at initial time and physical end-points
- \( C = (\square \ \square)_{0 \times 2} \) at final time
\[ \nabla^T(\beta u) = f \]

My first-order form:

\[
\beta_1 \frac{\partial u}{\partial x} + \beta_2 \frac{\partial u}{\partial y} + (\nabla^T \beta) u = f
\]

Examples:

- Circular advection: \( \beta(x, y) = \left( \begin{array}{c} -y \\ x \end{array} \right)^T / \| (x, y) \| \)

- 1+1 advection: \( \beta = \left( \begin{array}{c} 1 \\ 1 \end{array} \right)^T \)

Boundary equation (\( \nu \) is normal to boundary)

- If \( \beta^T \nu < 0 \) then \( C = \left( \begin{array}{c} 1 \end{array} \right) \)

- If \( \beta^T \nu > 0 \) then \( C = \left( \begin{array}{c} \Box \end{array} \right)_{0 \times 1} \)
PDE formulation

Elastic equation

Interior equation

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
D_{11} & 0 & 0 & 0 & 0 \\
D_{12} & 0 & 0 & 0 & 0 \\
0 & D_{33} & 0 & 0 & 0
\end{pmatrix} \frac{\partial w}{\partial x} + 
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & D_{12} & 0 & 0 & 0 \\
0 & D_{22} & 0 & 0 & 0 \\
D_{33} & 0 & 0 & 0 & 0
\end{pmatrix} \frac{\partial w}{\partial y} + 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix} w =
\begin{pmatrix}
F_1 \\
F_2 \\
0 \\
0 \\
0
\end{pmatrix}
\]

where

\[
w =
\begin{pmatrix}
u_1 \\
u_2 \\
D_{11} \frac{\partial u_1}{\partial x} + D_{12} \frac{\partial u_2}{\partial y} \\
D_{12} \frac{\partial u_1}{\partial x} + D_{22} \frac{\partial u_2}{\partial y} \\
D_{33} \left( \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)
\end{pmatrix}
\]
Boundary equation:

- Displacement condition: $C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

- Traction condition: $C = \begin{pmatrix} 0 & 0 & \nu_1 & 0 & \nu_2 \\ 0 & 0 & 0 & \nu_2 & \nu_1 \end{pmatrix}$

where $\nu = (\nu_1 \nu_2)^T$ is the normal to the boundary.

FYI: $D_{i,j}$ depend on elastic material parameters (e.g., Young’s modulus and Poisson’s ratio)
PDE formulation | Linearized stationary Navier–Stokes

\[-\nabla p + \nu \nabla^T \nabla u + (b^T \nabla)u + (u^T \nabla)b = f_1\]
\[\nabla^T u = f_2\]

In first-order form

\[
\begin{pmatrix}
b_1 & 0 & -1 & \nu & 0 & 0 & 0 \\
0 & b_1 & 0 & 0 & 0 & \nu & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\frac{\partial w}{\partial x} + \begin{pmatrix}
b_2 & 0 & 0 & 0 & \nu & 0 & 0 \\
0 & b_2 & -1 & 0 & 0 & 0 & \nu \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\frac{\partial w}{\partial y} + \begin{pmatrix}
\frac{\partial b_1}{\partial x} & \frac{\partial b_1}{\partial y} & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial b_2}{\partial x} & \frac{\partial b_2}{\partial y} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}w = \begin{pmatrix}
f_{1,1} \\
f_{1,2} \\
f_2 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
Unknowns

\[ w = \begin{pmatrix} u \\ p \\ \nabla u_1 \\ \nabla u_2 \end{pmatrix} \]

Boundary equations:

- Flow on walls: \[ C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

- Pressure on inlets and outlets: \[ C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \]
\[
\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f
\]
Bi-harmonic in first-order form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\frac{\partial w}{\partial x} +
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\frac{\partial w}{\partial y} +
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
w =
\begin{pmatrix}
f \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
Unknowns

\[ w = \begin{pmatrix} u(x) & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} & \frac{\partial^3 u}{\partial x^3} & \frac{\partial^3 u}{\partial x^2 \partial y} & \frac{\partial^3 u}{\partial y^3} \end{pmatrix}^T \]

Boundary equations

\[ C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu_1 & \nu_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]
**PDE formulation Div–Curl**

\[
\nabla^T A u = \rho \quad \text{(div)}
\]

\[
\nabla^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u = \omega \quad \text{(curl)}
\]

In first-order form

\[
\begin{pmatrix}
A_{11} & A_{12} \\
0 & 1
\end{pmatrix} \frac{\partial u}{\partial x} + \begin{pmatrix}
A_{21} & A_{22} \\
-1 & 0
\end{pmatrix} \frac{\partial u}{\partial y} + \left( \frac{\partial A_{11}}{\partial x} + \frac{\partial A_{21}}{\partial y} \quad \frac{\partial A_{12}}{\partial x} + \frac{\partial A_{22}}{\partial y} \right) u = \begin{pmatrix} \rho \\ \omega \end{pmatrix}
\]

Boundary equations (well–posed: Auchmuty–Alexander)

- \( \partial \Omega = \Gamma_N \cup \Gamma_T \) and \( \Gamma_N \cap \Gamma_T = \emptyset \) and \( \Gamma_N, \Gamma_T \) are connected non-empty sets
- \( C = \nu^T A \) on \( \Gamma_N \)
- \( C = \tau^T \) on \( \Gamma_T \)

If \( A \) is dis-continuous, the interface continuity conditions become:

\[
\begin{pmatrix}
\nu^T A_+ \\
\tau^T
\end{pmatrix} u_+ = \begin{pmatrix}
\nu^T A_- \\
\tau^T
\end{pmatrix} u_- = v
\]

- Code outputs this common value on the interface
- Can be modified to make it inhomogenous
Constant coeff. div-curl on:

\[ u = \begin{pmatrix} (1 + x^2 + y^2)^{-1} \\ x^2 - 2y^2 + xy - x + 1 \end{pmatrix} \]

Arrows: normal or tangential boundary condition

Grid spacing \( h \approx 0.08 \)

Accuracy: about 10 digits
Constant coeff. div-curl on:

\[ u = \begin{pmatrix} \frac{1 + x^2 + y^2}{x^2 - 2y^2 + xy - x + 1} \\ x^2 - 2y^2 + xy - x + 1 \end{pmatrix} \]

Arrows: normal or tangential boundary condition

Grid spacing \( h \approx 0.08 \)

Accuracy: about 10 digits
Tests  Div-curl  2 patch L–shape

Constant coeff. div-curl on:

\[
\begin{pmatrix}
(1 + x^2 + y^2)^{-1} \\
x^2 - 2y^2 + xy - x + 1
\end{pmatrix}
\]

Arrows: normal or tangential boundary condition

Grid spacing \( h \approx 0.1 \)

Accuracy: about 8 digits
Constants coeff. div-curl on:

Arrows: normal or tangential boundary condition

\[
\begin{pmatrix}
(1 + x^2 + y^2)^{-1} \\
x^2 - 2y^2 + xy - x + 1
\end{pmatrix}
\]

Grid spacing \( h \simeq 0.08 \)
Accuracy: about 11 digits
Constant coeff. div-curl on:

\[
\begin{pmatrix}
(1 + x^2 + y^2)^{-1} \\
x^2 - 2y^2 + xy - x + 1
\end{pmatrix}
\]

Arrows: normal or tangential boundary condition

Grid spacing \( h \approx 0.3 \)

Accuracy: about 8 digits
Constant coeff. div-curl on:

\[
u = \begin{pmatrix} \frac{1}{(1 + x^2 + y^2)^{-1}} \\ \frac{x^2 - 2y^2 + xy - x + 1}{x^2} \end{pmatrix}
\]

Only normal boundary condition!

Grid spacing \( h \approx 0.3 \)

Accuracy: about 6 digits
Constant coeff. div-curl on:

\[
\begin{pmatrix}
(1 + x^2 + y^2)^{-1} \\
 x^2 - 2y^2 + xy - x + 1
\end{pmatrix}
\]

Arrows: normal or tangential boundary condition

Grid spacing \( h \approx 0.5 \)

Accuracy: about 6 digits
Constant coeff. div-curl on:

\[
\mathbf{u} = \begin{pmatrix}
(1 + x^2 + y^2)^{-1} \\
x^2 - 2y^2 + xy - x + 1
\end{pmatrix}
\]

Arrows: normal or tangential boundary condition

Grid spacing \( h \approx 0.1 \)

Accuracy: about 8 digits
Tests Div-curl Exterior of car

Constant coeff. div-curl on:

\[ u = \begin{pmatrix} \frac{1}{1 + x^2 + y^2} \\ x^2 - 2y^2 + xy - x + 1 \end{pmatrix} \]

Normal boundary condition on car boundary
Tangential boundary condition on outside boundary

Grid spacing \( h \approx 0.6 \)
Accuracy: about 9 digits
Tests Poisson 2 patch rectangle

■ Domain: 

Grid spacing $h \approx 0.08$

- Accuracy $4 \times 3$ formulation: about 8 digits
- Accuracy $3 \times 3$ formulation: about 8 digits

■ Domain: 

Grid spacing $h \approx 0.6$

- Accuracy $4 \times 3$ formulation: about 4 digits
- Accuracy $3 \times 3$ formulation: about 4 digits
Tests Poisson $3 \times 3$

- Domain: 
  Grid spacing $h \approx 0.08$
  Accuracy: about 8 digits

- Domain: 
  Grid spacing $h \approx 0.1$
  Accuracy about 6 digits

- Domain: 
  Grid spacing $h \approx 0.08$
  Accuracy about 7 digits
Tests Wave equation

Constant coefficient wave equation

Domain:  

Grid spacing $h \approx 0.08$

Accuracy about 7 digits

Domain:  

Grid spacing $h \approx 0.08$

Accuracy about 7 digits
Constant coefficient heat equation

Domain:

Grid spacing $h \approx 0.08$
Accuracy about 8 digits
Tests Bi-harmonic equation

■ Domain:

Grid spacing $h \approx 0.08$
Accuracy about 6 digits

■ Domain:

Grid spacing $h \approx 0.6$
Accuracy about 3 digits

Grid spacing $h \approx 0.4$
Accuracy about 4 digits
Tests Linear advection

Domain: 

Grid spacing $h \approx 0.08$

Type: Circulating field
Accuracy about 9 digits

Type: $1+1$
Accuracy about 9 digits
• Constant coefficient div-curl

• Domain is $[-2, 2] \times [-1, 1]$ with a slit on $[-1, 1] \times \{0\}$

• Grid spacing $h \approx 0.1$

• Let $v(z) = (z^2 - 1)^{3/2}$ with branch cut on slit

$$u = \begin{pmatrix} v_{\text{Imag}} \\ v_{\text{Real}} \end{pmatrix}$$

• $u_1$ is dis-continuous across slit, but $u_2$ is continuous

• Poles at $(-1, 0)$ and $(1, 0)$

• Single normal boundary condition on slit
  
  – Accuracy about 5 digits

• Double tangential boundary condition on slit
  
  – Accuracy about 4 digits
Tests High-frequency Helmholtz

\[ \nabla^T \nabla u + 10^8 u = f \]

Domain: 🟢 🟠

Grid spacing \( h \approx 0.08 \)

Accuracy about 7 digits
Domain:

Young’s modulus: 1
Poisson’s ratio: 0.25
Displacement boundary condition on left vertical edge
Grid spacing $h \approx 0.08$
Accuracy about 9 digits
Tests  Linearized stationary Navier–Stokes

- Domain: 

Base flow: \((x + y \ xy)\)

True solution: \((y(2 - y) - 1 \quad (1 + x^2 + y^2)^{-1} \quad 1 - x)\)

Pressure boundary condition on left and right vertical edge

Flow boundary condition on top and bottom

Grid spacing \(h \simeq 0.08\)

Accuracy about 6 digits
Tests | Poisson in polar coordinates

Type: $3 \times 3$ formulation

- **Domain:**

  Singularity on left edge
  Dirichlet boundary conditions
  Grid spacing $h \approx 0.08$

- **Solution** $(1 + x^2 + y^2)^{-1}$
  Accuracy about 8 digits

- **Solution** $\text{Real}(z^{5/2})$
  Accuracy about 6 digits
Type: $3 \times 3$ formulation

■ Domain:
Third patch behind first two patches
Vertical edges are distinct
Solution Real($z^{5/2}$)
Top edges share boundary data
Bottom edges share boundary data (periodic)
Vertical edges do not share boundary data

- Grid spacing $h \approx 0.15$
  Accuracy about 3 digits

- Grid spacing $h \approx 0.1$
  Accuracy about 5 digits
Solution in all cases is suitably modified $(1 + x^2 + y^2)^{-1}$

- Domain:  
  Div–curl coeff. $A$ is 1 on first patch, 2 on second patch  
  Grid spacing $h \approx 0.08$  
  Accuracy about 10 digits

- Domain:  
  Div–curl coeff. $A$ is 1 on first patch, 2 on second patch, 3 on third patch  
  Grid spacing $h \approx 0.08$  
  Accuracy about 10 digits

- Domain:  
  Div–curl coeff. $A$ is 1 on first patch, 2 on second patch  
  Grid spacing $h \approx 0.1$  
  Accuracy about 8 digits
Tests | Div–curl | Self-consistency

- Domain: 
  - Tangential on right edge with value 1
  - Normal on left edge with value 0
  - Normal on top edge with value $x$
  - Normal on bottom edge with value $-x$
  - Grid spacing $h_1 \approx 0.077$ and $h_2 \approx 0.071$
  - Two solutions agreed to about 4 digits
\[ \nabla^T \nabla u - u = f \]

- **Solution:** \((1 + 10(x - y^2)^2)^{-1}\)
  - Singularity is on a parabola

- **Domain:** Circle of diameter 1

<table>
<thead>
<tr>
<th>Grid spacing</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2E-3</td>
</tr>
<tr>
<td>0.075</td>
<td>3E-4</td>
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<tr>
<td>0.05</td>
<td>4E-5</td>
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<tr>
<td>0.0375</td>
<td>1E-5</td>
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<tr>
<td>0.025</td>
<td>2E-6</td>
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</tbody>
</table>
Tests | Div–curl | Half-circle plus rectangle

Domain:

<table>
<thead>
<tr>
<th>Grid spacing</th>
<th>Digits of accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>3</td>
</tr>
<tr>
<td>0.2</td>
<td>4</td>
</tr>
<tr>
<td>0.1</td>
<td>8</td>
</tr>
</tbody>
</table>
Proof

Quick outline

- Keep number of patches fixed
- Assume infinite order polynomials on each patch (kernel or RBF approach)
- Compactness argument based on uniform bound on Sobolev norm
- Limit is a continuous function that satisfies PDE inside patches and on boundaries
- If PDE theory says this is a classical solution we are done
- With lot more effort we can also look at finite-order polynomials on each patch
Summary

• Golomb–Weinberger MSN technique for PDEs
• First-order formulation

Future Work

• Realistic tests with domain specialists
  – Send us your 2D problem
  – Use our code
• Eigenvalue problems
• Nonlinear problems
• 3D
Thank you!