Resurrecting Equi-Spaced Polynomial Interpolation

A distribution agnostic approach

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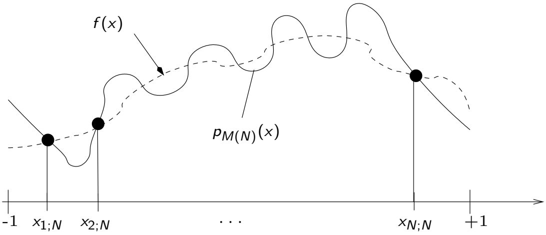
February 2008

Outline

Interpolation



Given

- N points $-1 \leq x_{1;N} < x_{2;N} < \cdots < x_{N;N} \leq 1$,
- samples of a function f: $f(x_{i;N})$,

find a polynomial $p_{M(N)}$ of degree $M(N)$ such that

- it interpolates: $p_{M(N)}(x_{i;N}) = f(x_{i;N})$, for $1 \leq i \leq N$,
- and approximates: $\lim_{N\uparrow\infty} p_{M(N)}(x) = f(x)$,

if x is a limit point of the x_N.

$M(N) = N - 1$

- Picking a polynomial of degree $N-1$ is the most straightforward.
- Unfortunately it lacks the approximation property in most situations.
- Runge's phenomenon
  - If $x_{i;N}$ is equi-spaced
  - $f(x) = (1+100x^2)^{-1}$

  Then p_{N-1} diverges from f. Note: f is analytic on the real line.
- Marcinkiewicz and Grünwald. If the $x_{i;N}$ are the zeros of the N-th degree Chebyshev polynomial, then there is a continuous f for which p_{N-1} diverges. Such an f has a modulus of continuity ω that does not satisfy the Dini-Lipschitz condition

$$\lim_{N\uparrow\infty} \omega\left(\frac{1}{N}\right) \log N = 0.$$

Current Practice

If possible use Chebyshev zeros.

- Current sampling technology prefers equi-spaced.
- In non-rectangular domains optimal sample point placement is more difficult.

Solution: Use low-order splines.

However, there is more in the theory, old and recent ...

Don't Preserve Polynomials

- **Féjer**: Use $M(N) = 2N - 1$, with zero derivative conditions

$$p'_{2N-1}(x_{i;N}) = 0.$$

  - Converges for all continuous functions if x_N are zeros of Chebyshev polynomial!
  - **Berman**: Diverges for equi-spaced points (Runge phenomenon).
- **Bernstein**: Give up interpolation and use

$$p_{N-1}(x) = \frac{1}{2^{N-1}} \sum_{k=0}^{N-1} \binom{N-1}{k} f(x_{k+1;N}) (1+x)^k (1-x)^{N-1-k}.$$

Converges for all continuous functions at equi-spaced points!

A Negative Result

- **Lozinskii** and **Kharshiladza**. Let U_N linearly map f on $[-1, 1]$ to a polynomial of degree N. Then $U_N(f)$ cannot converge for all continuous functions in any $1 \leq p \leq \infty$ norm, if U_N preserves polynomials of degree $\leq N$.
- **Timan**. If a triangular weighting scheme is used to damp the coefficients, convergence can be recovered (we loose polynomial preservation and interpolation).
- However, one can do better ...

A Positive Result

Erdös and **Szabados**. For every $\epsilon > 0$,

- there exists a polynomial of degree $\frac{\pi}{2}N(1+\epsilon)$
- that interpolates the given continuous function at N equi-spaced points on $[-1,1]$,
- and these polynomials converge uniformly to f,
- with error that is of the same order of magnitude as that of the best approximating polynomial of degree $\frac{\pi}{2}N(1+\frac{\epsilon}{3})$.

However

- Proof is non-constructive
- Generalization requires sample points to be dense in $[-1,1]$

Our Approach

We propose

- linear approach to construct
- an interpolating polynomial
- converges at limit points of the sample points
- does not preserve polynomials
- no constraint on interpolation points
- works in higher dimensions
- any basis with nice Sobolev norm

Polynomial Preservation

- Not a good idea in theory & practice
- Do not want to recover a 1000 degree polynomial from a "straight" line

Choosing The Basis

Chebyshev polynomials

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2\,x T_n(x) - T_{n-1}(x) \end{aligned}$$

or

$$T_n(x) = \cos(n \cos^{-1} x), \qquad -1 \leq x \leq 1$$

We will represent our polynomials as

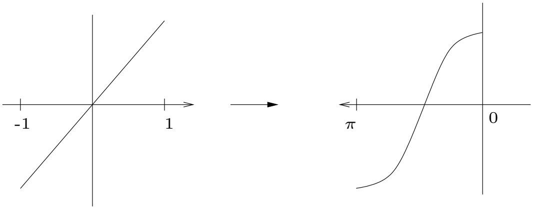
$$p_M = \sum_{n=0}^{M} a_n T_n$$

Gautschi. This is a reasonably stable representation.

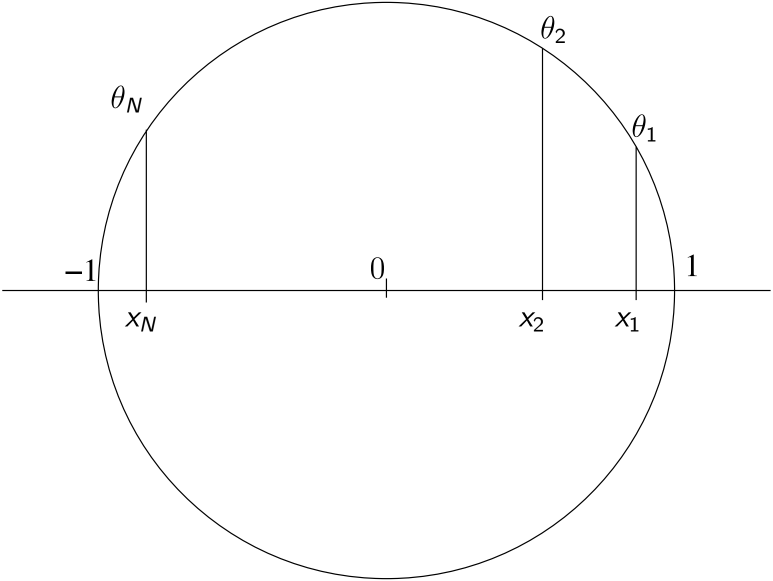
The Circle

- $\Gamma = $ unit circle in $\mathcal{R}^2$
- $\cos\theta = x$ maps Γ onto $[-1,1]$
- f smooth on $[-1,1] \equiv f \circ \cos$ smooth on Γ

$$
\begin{aligned}
\frac{d}{d\theta} f(\cos\theta) &= -f'(\cos\theta)\sin\theta \\
\frac{d^2}{d\theta^2} f(\cos\theta) &= f''(\cos\theta)\sin^2\theta - f'(\cos\theta)\cos\theta \\
\vdots &= \vdots
\end{aligned}
$$



Cosine Map



Fourier Cosine Series

$$
\begin{aligned}
f(x) = f(\cos\theta) &= \sum_{n=0}^{\infty} \hat{f}_n \cos(n\,\theta) \\
&= \sum_{n=0}^{\infty} \hat{f}_n T_n(x) \\
\hat{f}_0 &= \frac{1}{\pi} \int_0^{\pi} f(\cos\theta)\, d\theta \\
\hat{f}_n &= \frac{2}{\pi} \int_0^{\pi} f(\cos\theta) \cos(n\,\theta)\, d\theta
\end{aligned}
$$

Hölder Continuous

- α-Hölder norm of f:

$$\sup_{x,y\in[-1,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

- $C^{k+\alpha}(S) \equiv k$-th derivative is α-Hölder continuous on S
- If $f \in C^{k+\alpha}[-1,1]$ then

$$\hat{f}_n = O\left(\frac{1}{n^{k+\alpha}}\right)$$

- **Bernstein**. If $f \in C^\alpha[-1,1]$ with $s > \frac{1}{2}$, then $\hat{f}_n$ is absolutely summable.

Sobolev Spaces

If $f(\theta) = \sum_{n=0}^{\infty} \hat{f}_n \cos(n\theta)$, then

$$f'(\theta) = -\sum_{n=0}^{\infty} n\,\hat{f}_n \sin(n\theta)$$

$$f''(\theta) = \sum_{n=0}^{\infty} n^2 \hat{f}_n \cos(n\theta)$$

The Sobolev space H_s contains f if

$$\hat{f}_0^2 + \sum_{n=1}^{\infty} n^{2s} \hat{f}_n^2 < \infty.$$

Sobolev spaces and Hölder norms. $C^s \subseteq H_s \subseteq C^{s-\frac{1}{2}}$.

Damping Oscillations

- **Féjer**: oscillations in between interpolation nodes (Runge phenomenon) can be damped by controlling derivatives.
- Numerically difficult in spatial domain.

Key observations

- Easier for $(f \circ \cos)'$ than f'.
- Control derivative in L_2 rather than point-wise like Féjer.

Optimization Problem

Minimum Sobolev norm method: Choose $a_{M;n}$ to be the optimal solution of

$$\min_{a_{N;n}} a_{N;0}^2 + \sum_{n=1}^{M(N)} n^{2s} a_{N;n}^2$$

such that

$$\sum_{n=0}^{M(N)} a_{N;n} T_n(x_{i;N}) = f(x_{i;N}), \qquad 1 \le i \le N,$$

with $s > \frac{1}{2}$.

Recommendations

- $s = 1.5$
- $M(N) = 1.5 \left\lceil \frac{\pi}{\min_{i \neq j} |\cos^{-1}(x_{i;N}) - \cos^{-1}(x_{j;N})|} \right\rceil$

Convergence

Theorem

- Let x_N be a sequence of N distinct points in $[-1, 1]$.
- Let x be a limit point of the x_N.
- Let $f \in H_s[-1, 1]$ with $s > \frac{1}{2}$.
- Let c be a constant which depends only on x_N and s.
- Let

$$M(N) = c \left\lceil \frac{1}{\min_{i \neq j} |\cos^{-1}(x_{i;N}) - \cos^{-1}(x_{j;N})|} \right\rceil .$$

- Let $p_{M(N)}$ denote the optimal Sobolev s-norm interpolating polynomial for f at x_N.

Then, c can be chosen such that

$$\lim_{N \uparrow \infty} p_{M(N)}(x) = f(x),$$

for all $f \in H_s[-1, 1]$.

Convergence . . .

- Proof generalizes to higher dimensions: $[-1, 1]^d$
- Proof generalizes to other bases with suitable Sobolev p-norms
- Convergence is independent of point distribution
- H_s for $s \leq \frac{1}{2}$ contains dis-continuous functions
- Proof is very technical; uses Riesz–Thorin interpolation theorem and Marcinkiewicz–Zygmund type inequalities
- Draft tutorial in one dimension is available
- Much simpler to ignore tightness of $M(N)$ and concentrate on point distribution independent convergence

Convergence for $M(N) = \infty$ and $s > \frac{3}{2}$

Theorem

Let p_N denote the optimal infinite order polynomial that interpolates $f \in H_s[-1, 1]$ with $s > \frac{3}{2}$ at the given N distinct points $-1 \leq x_{i;N} \leq 1$ for $1 \leq i \leq N$. Let x be a limit point for the x_N. Then $\lim_{N \uparrow \infty} p_N(x) = f(x)$.

- Convergence is independent of point distribution.
- First derivative of f must be Hölder continuous with exponent greater than $\frac{1}{2}$.
- If $s = 2$ there is an exact computational procedure to find $a_{N;n}$.
- There is an effective truncation procedure; but interpolation property will become approximate rather than exact.

Proof

- The Fourier Cosine series is a feasible solution. Hence

$$a_{N;0}^2 + \sum_{n=1}^{\infty} n^{2s} a_{N;n}^2 \le \hat{f}_0^2 + \sum_{n=1}^{\infty} n^{2s} \hat{f}_n^2 < \infty.$$

- By Cauchy–Scwartz

$$\begin{aligned} \sum_{n=1}^{\infty} n|a_{N;n}| &= \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} n^s |a_{N;n}| \\ &\le \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{2(s-1)}}} \sqrt{\sum_{n=1}^{\infty} n^{2s} a_{N;n}^2} \\ &< \infty, \end{aligned}$$

if $s > \frac{3}{2}$.

Proof ...

- $\|(p_N \circ \cos)'\|_\infty$ is uniformly bounded
- $\|p_N \circ \cos\|_\infty$ is uniformly bounded
- **Ascoli–Arzelà**: there is a uniformly converging sub-sequence
- For some sub-sequence p_{N_j}

$$\lim_{j\uparrow\infty} \|p_{N_j} - p\|_\infty = 0$$

- For some sub-sub-sequence $x_{N_{j_k}}$

$$\lim_{k\uparrow\infty} x_{i_k;N_{j_k}} = x$$

- p is continuous; f is continuous

Proof ...

$$
\begin{aligned}
|p(x) - f(x)| &= \lim_{k\uparrow\infty} |p(x_{i_k;N_{j_k}}) - f(x_{i_k;N_{j_k}})| \\
&= \lim_{k\uparrow\infty} |p(x_{i_k;N_{j_k}}) - p_{N_{j_k}}(x_{i_k;N_{j_k}})| \\
&\leq \lim_{k\uparrow\infty} \|p - p_{N_{j_k}}\|_\infty \\
&= 0
\end{aligned}
$$

- There is nothing special about the sub-sequence p_{N_j} and its limit point p
- Therefore, every limit point of p_N must satisfy $p(x) = f(x)$
- In other words, $\lim_{N\uparrow\infty} p_N(x) = f(x)$

QED

Convergence for $M = \infty$ and $s > \frac{1}{2}$

- Only a little more effort ...
- Exact computational procedure for $s = 1$
- A lot more effort, but still elementary analysis, can handle $M = O(N^2)$
- Handling $M = O(N)$ is very technical
- Extendable to piece-wise smooth functions, if points of discontinuity are not limit points
- In practice the *opposite* works very well

Runge Phenomenon Banished

- $f(x) = (1 + 100x^2)^{-1}$
- N equi-spaced points on $[-1, 1]$ excluding boundary
- $M(N) = 2N$
- Larger s is better in this case, but the condition number of the constraint equations also increases

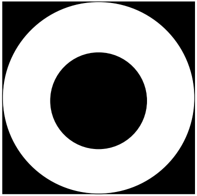
Maximum point-wise error between f and the interpolating polynomial:

N	$s = 1.5$	$s = 2.5$
15	$7 \cdot 10^{-2}$	$7 \cdot 10^{-2}$
30	$2 \cdot 10^{-2}$	$2 \cdot 10^{-2}$
60	$8 \cdot 10^{-4}$	$3 \cdot 10^{-4}$
120	$3 \cdot 10^{-4}$	$4 \cdot 10^{-6}$
240	$1 \cdot 10^{-4}$	$7 \cdot 10^{-7}$
480	$3 \cdot 10^{-5}$	$1 \cdot 10^{-7}$
960	$8 \cdot 10^{-6}$	$1 \cdot 10^{-8}$

Difficult 2D Interpolation

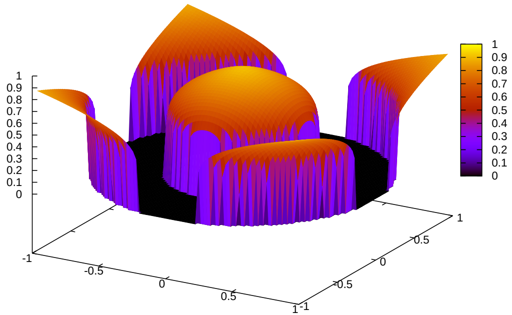
- $f(r, \theta) = (|r - 0.5|\,|r - 1|)^{\frac{1}{8}}$
- Domain is black region in figure below
- Curved line singularities on multiply connected domain
- 156 equi-spaced samples in domain
- 1929 2D Chebyshev polynomials with $s = 1.5$

Domain

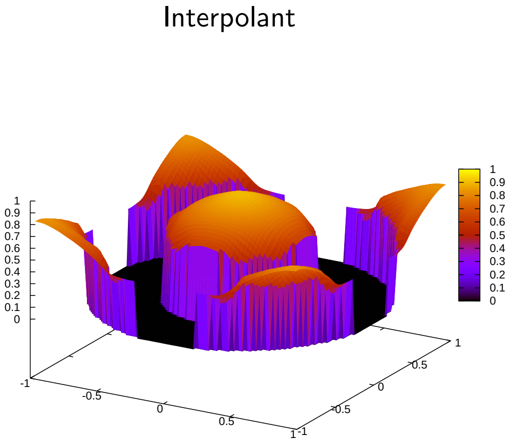


Difficult 2D Interpolation . . .

Original function



Difficult 2D Interpolation ...



Rapid Local Convergence

- 8th-root singularity is not covered by our theorem
- In 1D rate of convergence at a point is only based on the local smoothness of the function
- $f(x) = |x|^{\frac{1}{8}}$ on $[-1, 1]$.
- N equi-spaced samples; $M(N) = 6N$ and $s = 1.5$
- Errors between f and interpolant at $x = 0$ and $x = 0.5$

N	$x = 0$	$x = 0.5$
16	$7 \cdot 10^{-1}$	$7 \cdot 10^{-5}$
32	$6 \cdot 10^{-1}$	$3 \cdot 10^{-6}$
64	$6 \cdot 10^{-1}$	$1 \cdot 10^{-6}$
128	$5 \cdot 10^{-1}$	$2 \cdot 10^{-7}$
256	$5 \cdot 10^{-1}$	$2 \cdot 10^{-7}$
512	$5 \cdot 10^{-1}$	$4 \cdot 10^{-8}$
1024	$4 \cdot 10^{-1}$	$2 \cdot 10^{-8}$

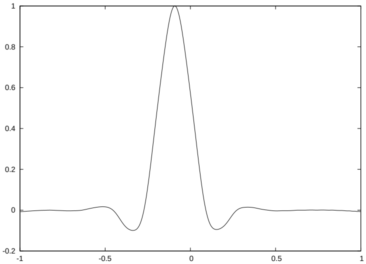
- Seems to be true in higher dimensions too

Interpolatory Kernels

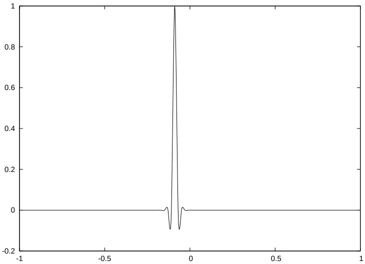
Due to linearity

$$p_{M(N)}(x) = \sum_{i=1}^{N} f(x_{i;N}) l_{i;N;M(N)}(x)$$

$l_{i;N;M(N)}$ is a polynomial of degree M that depends on x_N and s.



$l_{5;10;60}$



$l_{46;100;600}$

Speed

- In 1D there are traditional “matrix–free” methods
- We have fast algorithms instead

In Matrix Language

$$V_{N;M}(x_N) = \begin{pmatrix} 1 & \cos(\theta_{1;N}) & \cos(2\theta_{1;N}) & \cdots & \cos(M\theta_{1;N}) \\ 1 & \cos(\theta_{2;N}) & \cos(2\theta_{2;N}) & \cdots & \cos(M\theta_{2;N}) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & \cos(\theta_{N;N}) & \cos(2\theta_{N;N}) & \cdots & \cos(M\theta_{N;N}) \end{pmatrix}$$

$$D_s = \text{diag}\left(1^s \quad 2^s \quad 3^s \quad \cdots\right)$$

Optimization problem

$$\min_{a_N} \|D_s a_N\|_2, \qquad \text{such that} \qquad V_{N;M}(x_N)a_N = f(x_N)$$

Optimal solution

$$a_N = D_s^{-1}(V_{N;M}(x_N)D_s^{-1})^\dagger f(x_N)$$

Optimal polynomial

$$p_M(x) = V_{N;M}(x)D_s^{-1}(V_{N;M}(x_N)D_s^{-1})^\dagger f(x_N)$$

Matrix Structure

$$(V_N D_s^{-1})^\dagger = D_s^{-1} V_N^T (V_N D_s^{-2} V_N^T)^{-1}$$

Displacement Structure

$$(V_N D_s^{-1})(D_s(I + Z^2)) - (2X_N)(V_N D_s^{-1})(D_s Z) = \text{rank } 2$$

Sequentially Semi-Separable Structure

Numerical rank of off-diagonal blocks of $V_{N;M} D_s^{-2} V_{N;M}^T$ with $s = 1.5$ for N equi-spaced points.

N	M	10^{-8}	10^{-10}
320	1,514	7	10
640	3,020	7	10
1,280	6,038	7	10
2,560	12,068	7	10

Exact SSS

If $M = \infty$ then $V_N D_s^{-2} V_N^T$ has exact Hankel ranks:

- $s = 1 \Rightarrow$ Hankel rank $= 3$
- $s = 2 \Rightarrow$ Hankel rank $= 5$

Summary

Minimum Sobolev 2-norm methods in Chebyshev basis solve the polynomial interpolation problem in $\mathcal{R}^d$, in a distribution agnostic manner with good convergence properties

Future Work

- Can work with other bases: for example, interpolatory wavelet expansions; compressed sensing?
- Shows rapid local convergence: proof? meshless methods?
- Works as singularity detector on incomplete samples: applications to signal and image processing
- Interpolation on manifolds using Laplace–Beltrami eigenfunctions
- Works for boundedly invertible operators: discretization of partial differential and integral equations

Thank You!